# MATH 213 - EXAM 1 VERSION 1 - SOLUTIONS

1. $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ $\quad \vec{b} = 3\vec{i} + 5\vec{j} - 4\vec{k}$

(a) $\vec{a}\cdot\vec{b} = (1)(3) + (2)(5) + (2)(-4) = 5$

$|\vec{a}| = (1+4+4)^{1/2} = 3 \quad |\vec{b}| = (9+25+16)^{1/2} = 5\sqrt{2}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{5}{15\sqrt{2}} = \frac{1}{3\sqrt{2}} \;//$$

(b) $\vec{a} = |\vec{a}|\left(\frac{\vec{a}}{|\vec{a}|}\right) = 3\left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) //$

(c) $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 3 & 5 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 5 & -4 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}\vec{k}$$

$$= -18\vec{i} + 10\vec{j} - \vec{k} \;//$$

(d) $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{(324+100+1)^{1/2}}{15\sqrt{2}} = \frac{\sqrt{425}}{15\sqrt{2}}$$

$$= \frac{5\sqrt{17}}{15\sqrt{2}} = \frac{\sqrt{17}}{3\sqrt{2}} \;//$$

(e) $$\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{5}{9}\vec{a} = \frac{5}{9}\vec{i} + \frac{10}{9}\vec{j} + \frac{10}{9}\vec{k} \;//$$

(f) $$\vec{b} = \text{proj}_{\vec{a}}(\vec{b}) + \left(\vec{b} - \text{proj}_{\vec{a}}(\vec{b})\right)$$

$$= \left(\frac{5}{9}\vec{i} + \frac{10}{9}\vec{j} + \frac{10}{9}\vec{k}\right) + \left(\frac{22}{9}\vec{i} + \frac{55}{9}\vec{j} - \frac{26}{9}\vec{k}\right) \;//$$

(g) $\vec{r}(t) = \langle 6,5,-1\rangle + t\langle 1,2,2\rangle$

$= \langle 6+t, 5+2t, -1+2t\rangle$ //

2. $\vec{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle$

(a) $\vec{r}\,'(t) = \langle 3\cos t, -5\sin t, 4\cos t\rangle$ (velocity)

$|\vec{r}\,'(t)| = (9\cos^2 t + 25\sin^2 t + 16\cos^2 t)^{1/2}$
$= (25\cos^2 t + 25\sin^2 t)^{1/2} = 5$ (speed)

$\vec{r}\,''(t) = \langle -9\sin t, -5\cos t, -4\sin t\rangle$ (acceleration)

(b) $|\vec{r}(t)| = (9\sin^2 t + 25\cos^2 t + 16\sin^2 t)^{1/2}$
$= (25\sin^2 t + 25\cos^2 t)^{1/2}$
$= 5$

$\therefore$ trajectory lies on a sphere of radius 5 centered at the origin

3. $\vec{r}(t) = t\,\vec{i} + \frac{1}{3}t^{3/2}\vec{j} + t\,\vec{k}$

(a) $\vec{r}\,'(t) = \vec{i} + \frac{1}{2}t^{1/2}\vec{j} + \vec{k}$

$|\vec{r}\,'(t)| = \left(1 + \frac{1}{4}t + 1\right)^{1/2} = (2 + t/4)^{1/2} = \frac{1}{2}(t+8)^{1/2}$

$\therefore \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{2t}{\sqrt{t+8}}\vec{i} + \dfrac{2t^{3/2}}{3\sqrt{t+8}}\vec{j} + \dfrac{2t}{\sqrt{t+8}}\vec{k}$ //

(b) $L = \int_0^4 |\vec{r}\,'(t)|\,dt = \int_0^4 \frac{1}{2}(t+8)^{1/2}\,dt$

$= \frac{1}{2}\cdot\frac{2}{3}(t+8)^{3/2}\Big|_0^4 = \frac{1}{3}\left(12^{3/2} - 8^{3/2}\right)$

$= \frac{8}{3}\left(3^{3/2} - 2^{3/2}\right)$ //

4. $\vec{a}(t) = \vec{j} - 32\vec{k}$

$$\vec{r}'(t) = t\vec{j} - 32t\vec{k} + \vec{c}$$

$$\vec{r}'(0) = 50\vec{i} + 50\sqrt{3}\,\vec{k} = \vec{c}$$

$$\vec{r}'(t) = 50\vec{i} + t\vec{j} + (50\sqrt{3} - 32t)\vec{k}$$

$$\vec{r}(t) = 50t\vec{i} + \tfrac{1}{2}t^2\vec{j} + (50\sqrt{3}t - 16t^2)\vec{k} + \vec{c}$$

$$\vec{r}(0) = \vec{0} = \vec{c} \quad \therefore \vec{c} = 0$$

$$\vec{r}(t) = 50t\vec{i} + \tfrac{1}{2}t^2\vec{j} + (50\sqrt{3}t - 16t^2)\vec{k}$$

# MATH 213 - EXAM 1 VERSION 2 - SOLUTIONS

1. $\vec{a} = 3\vec{i} + 5\vec{j} - 4\vec{k}$ $\quad \vec{b} = \vec{i} + \vec{j} - 3\vec{k}$

(a) $\vec{a} \cdot \vec{b} = (3)(1) + (5)(1) + (-4)(-3) = 3 + 5 + 12 = \underline{20}$

$|\vec{a}| = (9 + 25 + 16)^{1/2} = 5\sqrt{2}$ $\quad |\vec{b}| = (1 + 1 + 9)^{1/2} = \sqrt{11}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{20}{5\sqrt{2}\cdot\sqrt{11}} = \frac{4}{\sqrt{22}} \;//$$

(b) $\vec{u} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{5\sqrt{2}}(3\vec{i} + 5\vec{j} - 4\vec{k}) = \dfrac{3}{5\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j} - \dfrac{4}{5\sqrt{2}}\vec{k}$

or $\vec{u} = \dfrac{-\vec{a}}{|\vec{a}|} = \dfrac{-3}{5\sqrt{2}}\vec{i} - \dfrac{1}{\sqrt{2}}\vec{j} + \dfrac{4}{5\sqrt{2}}\vec{k}$

(c) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 5 & -4 \\ 1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & -4 \\ 1 & -3 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}\vec{k}$

$= -11\vec{i} + 5\vec{j} - 2\vec{k} \;//$

(d) Since $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ always, $\theta = \frac{\pi}{2} \;//$

(e) $\text{proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \frac{20}{11}(\vec{i}+\vec{j}-3\vec{k})$

$= \frac{20}{11}\vec{i} + \frac{20}{11}\vec{j} - \frac{60}{11}\vec{k}$ //

(f) $\vec{a} = \text{proj}_{\vec{b}}(\vec{a}) + (\vec{a} - \text{proj}_{\vec{b}}(\vec{a}))$

$= \left(\frac{20}{11}\vec{i} + \frac{20}{11}\vec{j} - \frac{60}{11}\vec{k}\right) + \left(\frac{13}{11}\vec{i} + \frac{35}{11}\vec{j} + \frac{16}{11}\vec{k}\right)$ //

2. $\vec{r}(t) = \langle 3+t^3, 2-4t^3, 1+6t^3\rangle$

(a) $\vec{r}'(t) = \langle 3t^2, -12t^2, 18t^2\rangle$ (velocity)

$|\vec{r}'(t)| = (9t^4 + 144t^4 + 324t^4)^{1/2} = t^2\sqrt{477}$ //
(speed)

$\vec{r}''(t) = \langle 6t, -24t, 36t\rangle$ (acceleration)

(b) $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle 3t^2, -12t^2, 18t^2\rangle}{t^2\sqrt{477}}$

$= \left\langle \frac{3}{\sqrt{477}}, \frac{-12}{\sqrt{477}}, \frac{18}{\sqrt{477}}\right\rangle$

Since $\vec{T}(t)$ is constant, motion is along a line in that direction.

3. $\vec{r}(t) = \cos(2t)\vec{i} + \sin(2t)\vec{j} + \frac{4}{3}t^{3/2}\vec{k}$

(a) $\vec{r}'(t) = -2\sin(2t)\vec{i} + 2\cos(2t)\vec{j} + 2t^{1/2}\vec{k}$

$|\vec{r}'(t)| = (4\sin^2(2t) + 4\cos^2(2t) + 4t)^{1/2}$

$= (4+4t)^{1/2} = 2(1+t)^{1/2}$

$\vec{r}''(t) = -4\cos(2t)\vec{i} - 4\sin(2t)\vec{j} + t^{-1/2}\vec{k}$ //

(b) $L = \int_1^2 2(1+t)^{1/2}\,dt = 2\cdot\frac{2}{3}(1+t)^{3/2}\Big|_1^2$

$= \frac{4}{3}(2^{3/2}-1)$ //.

4. $\vec{a}(t) = \vec{j} - 32\vec{k}$

$\vec{r}'(t) = t\vec{j} - 32t\vec{k} + \vec{c}$

$\vec{r}'(0) = 50\vec{i} + 50\sqrt{3}\,\vec{k} = \vec{c}$

$\vec{r}'(t) = 50\vec{i} + t\vec{j} + (50\sqrt{3} - 32t)\vec{k}$

$\vec{r}(t) = 50t\vec{i} + \frac{1}{2}t^2\vec{j} + (50\sqrt{3}t - 16t^2)\vec{k} + \vec{c}$

$\vec{r}(0) = \vec{0} = \vec{c} \quad \therefore \vec{c} = 0$

$\vec{r}(t) = 50t\vec{i} + \frac{1}{2}t^2\vec{j} + (50\sqrt{3}t - 16t^2)\vec{k}$