Quiz 13 today 14.2, 14.3

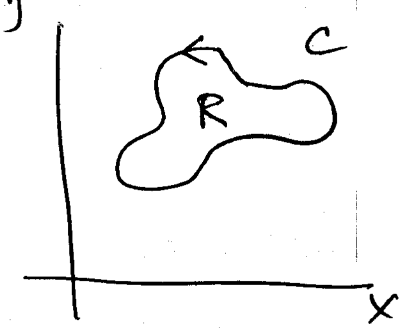
Final Exam – Thurs 5–10

$\sim\frac{1}{2}$ will cover topics after 2nd midterm.

$\sim\frac{1}{2}$ will cover earlier material

Green' Thm: $\vec{F} = \langle f, g \rangle$ $C: \vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$.
closed, simple curve.

① $\iint_R (\operatorname{curl}\vec{F} \cdot \vec{k})\, dA = \oint_C \vec{F} \cdot d\vec{r}$

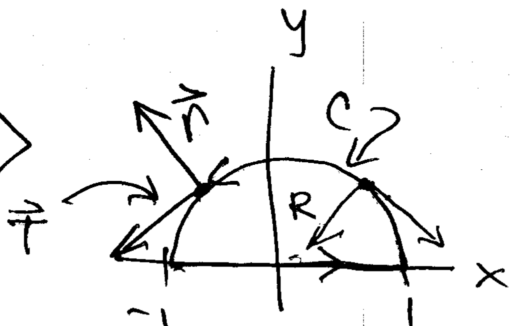


② $\iint_R \operatorname{div}\vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds$

These are like a version of FTC for vector fields.

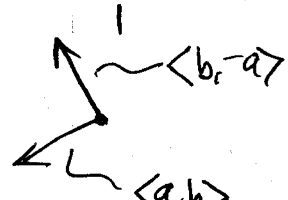
eg #32 p1008

$\int_C f\,dy - g\,dx$ $\langle f, g \rangle = \langle x^2, 2y^2 \rangle$



$C: \vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{T} \sim \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ $\vec{n} \sim \left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle$



Divergence form:

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \underset{\vec{F}}{\langle f, g \rangle} \cdot \underbrace{\frac{\left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle}{\cancel{|\vec{r}\,'(t)|}}}_{\vec{n}} \underbrace{\cancel{|\vec{r}\,'(t)|}\, dt}_{ds}$$

$$\int_a^b \left(f\,\frac{dy}{dt} - g\,\frac{dx}{dt}\right) dt \xrightarrow[\text{written}]{} \underline{\int_C f\,dy - g\,dx}$$

So $\displaystyle\iint_R \operatorname{div}\vec{F}\,dA \underset{\text{Greens Thm}}{=} \int_C f\,dy - g\,dx$ $\qquad \vec{F} = \langle f, g\rangle = \langle x^2, 2y^2\rangle$

$$\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2x + 4y.$$

$$\int_C f\,dy - g\,dx \overset{\text{Greens Thm}}{=} \iint 2x + 4y\,dA.$$

$$= \int_0^{\pi}\int_0^1 (2r\cos\theta + 4r\sin\theta)(r\,dr\,d\theta)$$

$$= \int_0^{\pi}\int_0^1 (2r^2\cos\theta + 4r^2\sin\theta)\,dr\,d\theta$$

14.5 Divergence and Curl.

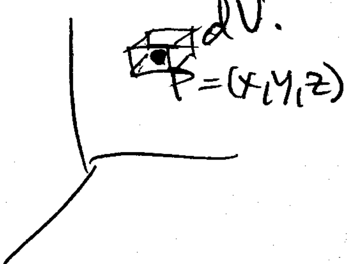
A. Divergence

$\vec{F} = \langle f, g, h \rangle \qquad \operatorname{div} \vec{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$

Symbolically: $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$

$\operatorname{div} \vec{F} = \nabla \bullet \vec{F} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle \bullet \langle f, g, h \rangle$

$(\operatorname{div} \vec{F})\, dV$ = rate at which fluid flows out of box at (x,y,z).



eg #12 $\vec{F} = \langle x^2yz, -xy^2z, -xyz^2 \rangle$

$\operatorname{div} \vec{F} = 2xyz + (-2xyz) + (-2xyz) = -2xyz$

#22 $\vec{F} = \langle x, y^2 \rangle \qquad \operatorname{div} \vec{F} = 1 + 2y$

$\operatorname{div} \vec{F}(-1,1) = 3 \qquad \operatorname{div} \vec{F}(-1,-1) = 1 - 2 = -1$

B. Curl

$\vec{F} = \langle f, g, h \rangle$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$$

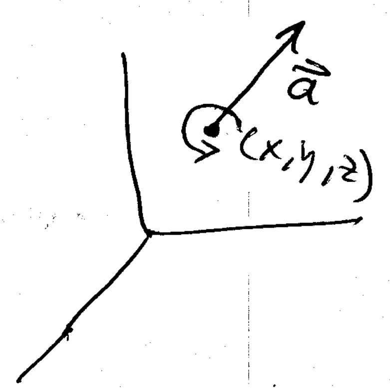
(a) 2 properties:

1) $\text{curl}(\nabla \varphi) = \nabla \times \nabla \varphi = \vec{0}$

$\text{curl}(\vec{F}) = \vec{0}$ if and only if $\vec{F}$ is conservative.

2) $\text{div}(\text{curl}\,\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$

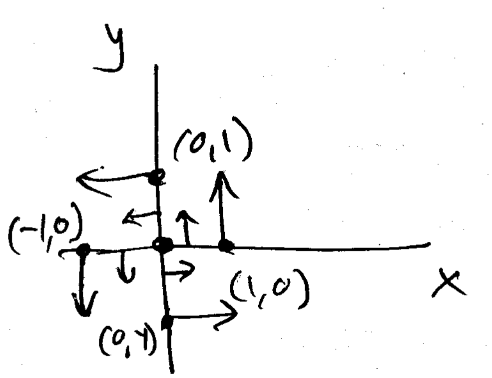
(b) Pure rotation vector field.



Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be fixed

$\vec{F} = \vec{a} \times \vec{r} = \vec{a} \times \langle x, y, z \rangle$

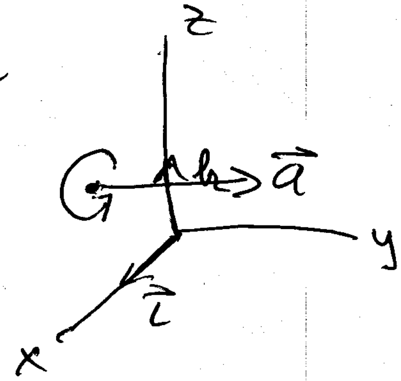
$\vec{a} = \langle 0, 0, 1 \rangle$ $\quad \vec{a} \times \langle x, y, z \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix}$



$= (-y\vec{i}) - (-x\vec{j}) = -y\vec{i} + x\vec{j} = \vec{F}$

$\vec{a} = \langle 0,1,0 \rangle \quad \vec{a} \times \langle x,y,z \rangle = z\vec{i} - x\vec{k}$

$\vec{a} = \langle 1,0,0 \rangle \quad \vec{a} \times \langle x,y,z \rangle = z\vec{i} - y\vec{j}$



Point: $\text{curl}(\vec{F}) = \text{curl}(\vec{a} \times \langle x,y,z \rangle) = 2\vec{a}$

eg $\vec{F} = \langle x^2yz, -xy^2z, -xyz^2 \rangle$

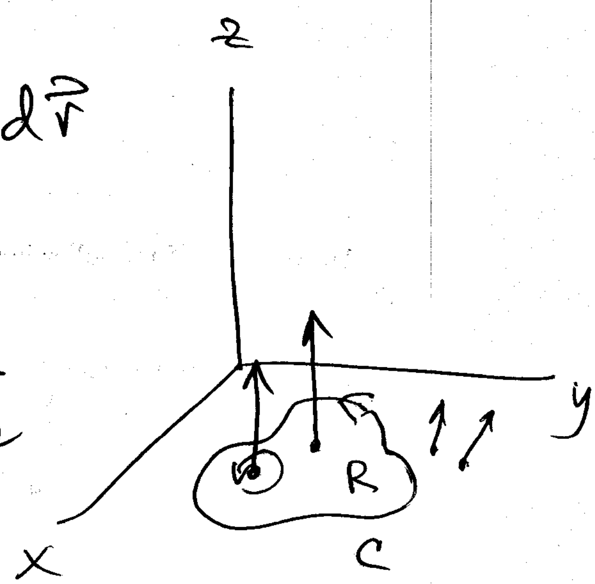
$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & -xy^2z & -xyz^2 \end{vmatrix} = \nabla \times \vec{F}$$

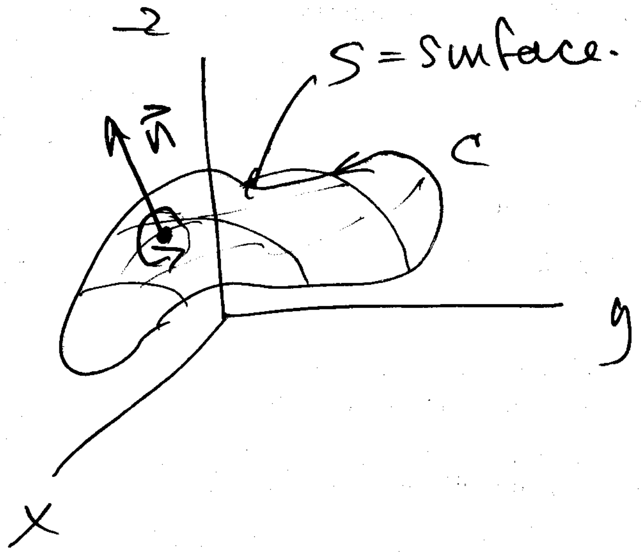
$$= (-xz^2 + xy^2)\,\vec{i} - (-yz^2 - x^2y)\,\vec{j} + (-y^2z - x^2z)\,\vec{k}$$

Greens Thm: $\vec{F} = \langle f, g \rangle$

$$\iint_R (\text{curl } \vec{F}) \cdot \vec{k} \, dA = \int_C \vec{F} \cdot d\vec{r}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = (f_y - g_x)\, \vec{k}$$

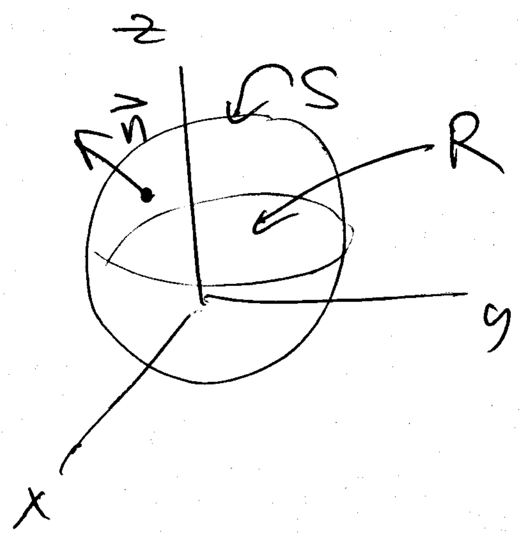




$(\text{curl } \vec{F}) \cdot \vec{n}$ = circulation on the surface

$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS = \int_C \vec{F} \cdot d\vec{r}$$

Stoke's Theorem.



$$\iiint_R \text{div } \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS$$

Divergence Theorem