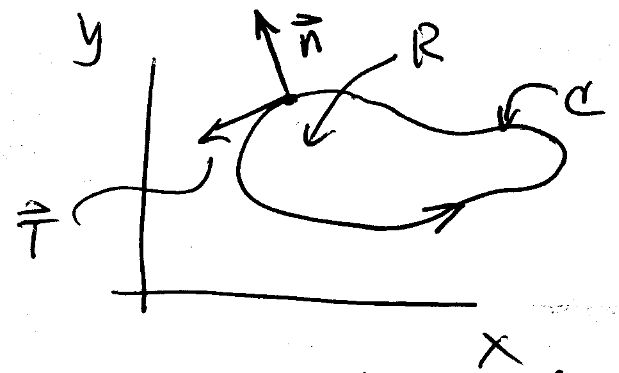
Quiz 13 will cover 14.2, 14.3

Green's Theorem.



$\vec{F}$ - vector field (think of a velocity field)

2 line integrals of $\vec{F}$ on C:

① circulation $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

② flux $\int_C \vec{F}\cdot\vec{n}\,ds$

Green's Thm relates ① and ② to the double integral of some "derivative" of $\vec{F}$ over R.

(a) Divergence of $\vec{F}$, $\text{div}(\vec{F})$.

if $\vec{F} = \langle f, g\rangle$ $\text{div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

Think of: $\nabla\vec{F} = \langle f_x, g_y\rangle$

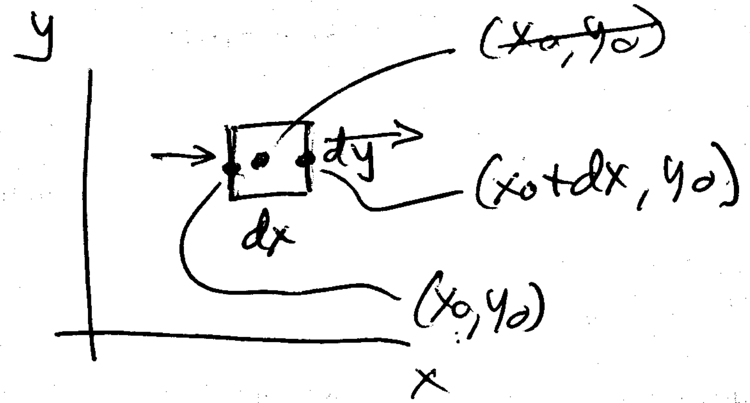
$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$ $\nabla\cdot\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle\cdot\langle f, g\rangle$

$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$.

$\boxed{\text{symbolically, } \text{div}(\vec{F}) = \nabla\cdot\vec{F}}$

Note: $\operatorname{div}\vec{F}$ is a scalar quantity, a function of (x, y).

Interpretation: $\operatorname{div}\vec{F}$ measures the tendency of the fluid to move away from a point.



$\vec{F} = \langle f, g \rangle$

In x-direction, measure (flow out) $-$ (flow in)

flow out: $f(x_0+dx, y_0)\,dy$

flow in: $f(x_0, y_0)\,dy$

Net flow: $\left(\dfrac{f(x_0+dx, y_0) - f(x_0, y_0)}{dx}\right) dy\,dx$

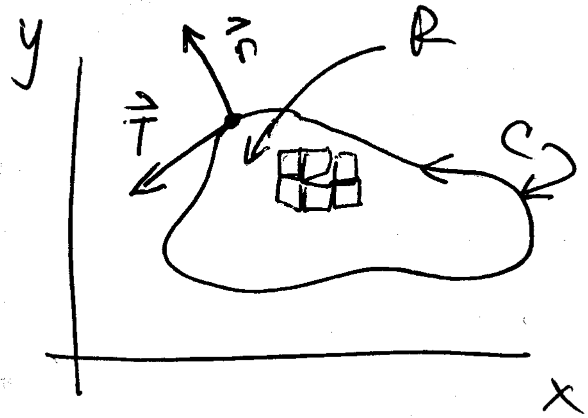
$$= f_x(x_0, y_0)\,dy\,dx = f_x(x_0, y_0)\,dA$$

In y-direction we get

Net flow $= f_y(x_0, y_0)\,dx\,dy = f_y(x_0, y_0)\,dA.$

Total net flow $= (f_x(x_0, y_0) + f_y(x_0, y_0))\,dA$

$$= (\operatorname{div}\vec{F})\,dA.$$



Total flow out of R is

$$\iint_R (\operatorname{div} \vec{F})\, dA = \int_C (\vec{F} \cdot \vec{n})\, ds$$ Green's Thm Part 1

Specifically: $\vec{F} = \langle f, g \rangle$.

$\operatorname{div} \vec{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$.

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

What is $\vec{n}$? $C: \vec{r}(t),\ a \le t \le b$, SAY $|\vec{r}\,'(t)| = 1$

$\vec{T} = \dfrac{\vec{r}\,'(t)}{\underbrace{|\vec{r}\,'(t)|}_{1}} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle$

$\vec{n} = \left\langle \dfrac{dy}{dt}, -\dfrac{dx}{dt} \right\rangle$ (if $|\vec{r}\,'(t)| = 1$).

In general, $|\vec{r}\,'(t)| \ne 1$ then unit normal is

$$\vec{n} = \frac{\left\langle \dfrac{dy}{dt}, -\dfrac{dx}{dt} \right\rangle}{|\vec{r}\,'(t)|}.$$

So $\displaystyle\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \frac{\langle f, g\rangle \cdot \langle \frac{dy}{dt}, -\frac{dx}{dt}\rangle}{\cancel{|\vec{r}\,'(t)|}} \cancel{|\vec{r}\,'(t)|}\, dt$

$$= \int_a^b \left(f \frac{dy}{dt} - g \frac{dx}{dt} \right) dt$$

We write : $\displaystyle\int_C f\, dy - g\, dx$.

Green #1 : $\displaystyle\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \int_C f\, dy - g\, dx$.

(b) Curl of $\vec{F}$.

$\text{curl}(\vec{F}) = \nabla \times \vec{F}$ (symbolically)

$\text{curl}(\vec{F})$ measures tendency of fluid to circulate around ~~(x,y)~~ a point.

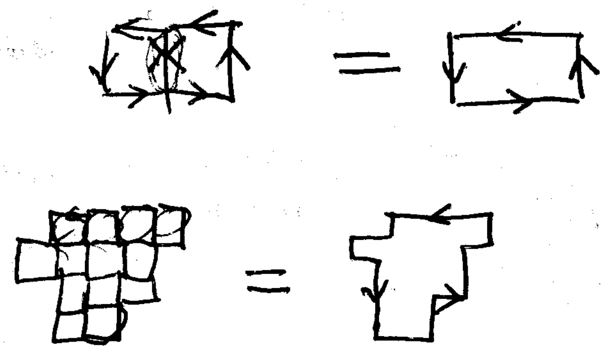
$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \vec{F} = \langle f, g, \cancel{h} \rangle = \langle f, g, 0 \rangle$

only 2-d vector fields.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix}$$

$$= \left(0 - \cancelto{0}{\frac{\partial g}{\partial z}}\right) \vec{i} - \left(0 - \cancelto{0}{\frac{\partial f}{\partial z}}\right) \vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \vec{k}$$

$$\operatorname{curl}(\vec{F}) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)(\vec{k})$$



$$\iint_R \left(\operatorname{curl}(\vec{F}) \cdot \vec{k}\right) dA = \int_C \vec{F} \cdot \vec{T}\, ds = \oint_C \vec{F} \cdot d\vec{r}.$$

Green's Thm Part 2.

More specifically:

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \int_C (f\, dx + g\, dy) = \int_C \vec{F} \cdot d\vec{r}$$

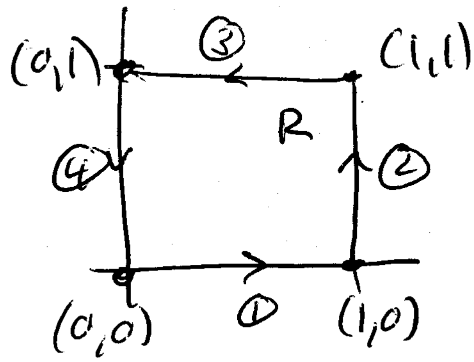
Green's #2

$\vec{r}(t) = \langle x(t), y(t) \rangle$

$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

$$\vec{F} \cdot \vec{r}\,'(t)\, dt$$
$$= \langle f, g \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$
$$= \left(f \frac{dx}{dt} + g \frac{dy}{dt}\right) dt$$
$$= f\, dx + g\, dy.$$

#12) $\vec{F} = \langle y, x \rangle$



$\text{curl}(\vec{F}) = (1-1)\vec{k} = \vec{0}$

$$\iint_R \text{curl}(\vec{F}) \cdot \vec{k}\, dA = \iint_R 0\, dA = 0.$$

Green's Thm says: $\iint_R (\text{curl}(\vec{F}) \cdot \vec{k})\, dA = \int_C \vec{F} \cdot d\vec{r}$

1:

$\vec{r}(t) = \langle t, 0 \rangle,\ 0 \le t \le 1.\quad \vec{r}\,'(t) = \langle 1, 0 \rangle$

$$\int_{1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 \langle 0, t \rangle \cdot \langle 1, 0 \rangle\, dt$$

$$= \int_0^1 0\, dt = 0.$$

2: $\vec{r}(t) = \langle 1, t \rangle,\ 0 \le t \le 1,\quad \vec{r}\,'(t) = \langle 0, 1 \rangle$

$$\int_{2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle t, 1 \rangle \cdot \langle 0, 1 \rangle\, dt = \int_0^1 dt = 1.$$

3: $\vec{r}(t) = \langle t, 1 \rangle,\ 0 \le t \le 1\quad \vec{r}\,'(t) = \langle 1, 0 \rangle$

$$\int_{3} \vec{F} \cdot d\vec{r} = \int_1^0 \langle 1, t \rangle \cdot \langle 1, 0 \rangle\, dt = -\int_0^1 dt = -1$$

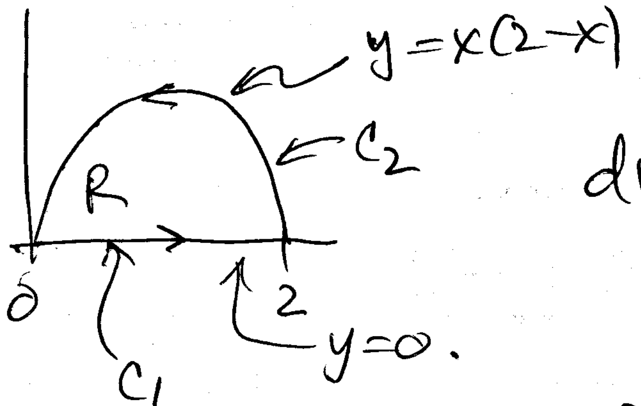
note right-to-left orientation of 3.

④: $\vec{r}(t) = \langle 0, 1-t \rangle$, $0 \le t \le 1$. $\vec{r}'(t) = \langle 0, -1 \rangle$

$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 \langle 1-t, 0 \rangle \cdot \langle 0, -1 \rangle \, dt = 0.$$

$$\therefore \int_{①} + \int_{②} + \int_{③} + \int_{④} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = 0.$$

eg #27) $\vec{F} = \langle 2xy, x^2 - y^2 \rangle$



$$\operatorname{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

$$= 2y - 2y = 0.$$

$$\therefore \iint_R \operatorname{div} \vec{F} \, dA = 0$$

Green's Theorem says: $\displaystyle\iint_R \operatorname{div} \vec{F} \, dA = \int_C \vec{F} \cdot \vec{n} \, ds$
$= 0.$

Verify this:

C_1: $\vec{r}(t) = \langle t, 0 \rangle$, $0 \le t \le 2$.

$\vec{r}'(t) = \langle 1, 0 \rangle$ $\quad \vec{n}(t) = \dfrac{\langle 0, -1 \rangle}{\| \langle 0, -1 \rangle \|} = \langle 0, -1 \rangle$

$\oint_{C_1}$

$$\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_0^2 \langle 2t\cdot 0, t^2-0^2\rangle\cdot\langle 0,-1\rangle\,dt$$

$$= \int_0^2 \langle 0, t^2\rangle\cdot\langle 0,-1\rangle dt = \int_0^2 -t^2\,dt = -\frac{1}{3}t^3\Big|_0^2 = \frac{-8}{3}.$$

C_2: $\vec{r}(t) = \langle t, t(2-t)\rangle$, $0 \le t \le 2$. Note orientation is reversed so try $\vec{r}(t) = \langle 2-t, (2-t)t\rangle$ $0 \le t \le 2$. (We have replaced t by $2-t$). This is OK.

$\vec{r}\,'(t) = \langle -1, 2-2t\rangle$ so $\vec{n}(t) = \dfrac{\langle 2-2t, 1\rangle}{\|\langle 2-2t,1\rangle\|}$ ← ignore

$$\int_{C_2} \vec{F}\cdot\vec{n}\,ds = \int_0^2 \langle 2t(2-t)^2, (2-t)^2-(2-t)^2t^2\rangle\cdot\langle 2-2t, 1\rangle\,dt$$

$$= \int_0^2 (2-t)^2\langle 2t, 1-t^2\rangle\cdot\langle 2-2t,1\rangle\,dt$$

$$= \int_0^2 (2-t)^2(4t-4t^2+1-t^2)\,dt$$

$$= \int_0^2 (4-4t+t^2)(1+4t-5t^2)\,dt$$

$$= \int_0^2 4-35t^2+24t^3-5t^4\,dt$$

$$= 4t \underset{+12t}{\vee} -\frac{35}{3}t^3+6t^4-t^5\Big|_0^2 = 8\overset{+24}{\cancel{1}} - \frac{8\cdot 35}{3}+96-32$$

(p8)

$$= \frac{8\cdot 36}{3} - \frac{8\cdot 35}{3} = \frac{8}{3} \;//$$

Therefore $\int_{C_1} + \int_{C_2} \vec{F} \cdot \vec{n}\, ds = -\frac{8}{3} + \frac{8}{3} = 0.$