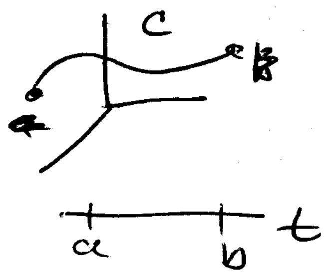
Quiz 12 Today 13.7

Vector Field $\vec{F}(x,y,z)$ or $\vec{F}(x,y)$

e.g. force fields or velocity fields.

Line integral.

① Scalar line integral: $\int_C f\,ds$.




$C: \vec{r}(t),\ a \le t \le b$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

② Vector fields: $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

Interpretation: work done by force field $\vec{F}$ moving object along $C$. OR circulation of fluid with velocity $\vec{F}$ along curve $C$.

$C: \vec{r}(t),\ a \le t \le b$ $\quad \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$.

③ $\int_C \vec{F}\cdot\vec{n}\,ds$ Interpretation: flux of fluid across $C$ if fluid has velocity $\vec{F}$.

$\vec{n}$: unit normal vector to $C$.

Conservative Vector Fields:

<u>Idea</u>: $\nabla\varphi$ is a vector field: $\nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

Q: Is every vector field of this form? NO

If $\vec{F} = \nabla\varphi$ then $\vec{F}$ is <u>conservative</u>.

① Can you tell easily if a v.f. $\vec{F}$ is conservative?

Yes: $\vec{F}(x,y) = f(x,y)\vec{i} + g(x,y)\vec{j}$

$\vec{F}$ is conservative $\Leftrightarrow$ $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$

$\vec{F}$ is conservative $\Leftrightarrow$ $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$

eg #14 $\vec{F} = \langle 2x^3 + xy^2, 2y^3 + x^2y \rangle$

Conservative? $\frac{\partial f}{\partial y} = 2xy$ $\frac{\partial g}{\partial x} = 2xy$ Yes.

This means $\vec{F} = \nabla \varphi$ for some $\varphi(x,y)$.

Can we find $\varphi$? Yes. How?

$\varphi_x = 2x^3 + xy^2 \longrightarrow \varphi = \frac{1}{2}x^4 + \frac{1}{2}x^2y^2 + \underline{g(y)}$

$\varphi_y = 2y^3 + x^2y \qquad \varphi_y = x^2y + g'(y)$

$$2y^3 + \cancel{x^2y} = \cancel{x^2y} + g'(y)$$

$$2y^3 = g'(y)$$

$$g(y) = \frac{1}{2}y^4 + C$$

$\therefore \varphi = \frac{1}{2}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^4 + C$.

#22) $\vec{F} = \langle yz, xz, xy \rangle$

$\quad\quad\quad\quad f \quad g \quad h$

Conservative? $f_y = z \quad g_x = z$ ✓

$f_z = y \quad h_x = y$ ✓ $\quad$ <u>Yes</u>

$g_z = x \quad h_y = x$ ✓

Find $\varphi$ : $\vec{F} = \nabla\varphi$ $\quad \varphi_x = yz \longrightarrow \varphi = xyz + r(y,z)$

$\varphi_y = xz \quad\quad \varphi_y = xz + r_y(y,z)$

$xz = xz + r_y(y,z)$

$\varphi_z = xy \quad\quad \varphi = xyz + s(z)$

$r_y(y,z) = 0$

$\varphi_z = xy + s'(z)$

$r(y,z) = s(z)$

$xy = xy + s'(z)$

$s'(z) = 0$

$s(z) = c$

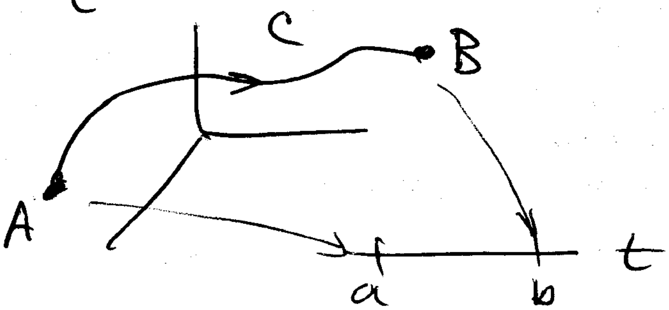
$\therefore \varphi = xyz + c$ //

② Line integrals of conservative v.f.s

$C: \vec{r}(t),\ a \le t \le b.\quad \vec{F} = \nabla\varphi$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A)$$

$B = \vec{r}(b),\ A = \vec{r}(a)$



Why? $\int_C \nabla\varphi \cdot d\vec{r} = \int_a^b \nabla\varphi(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$\nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle \quad \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\nabla\varphi(\vec{r}(t)) = \langle \varphi_x(x(t), y(t), z(t)), \varphi_y(x(t), y(t), z(t)), \varphi_z(x(t), y(t), z(t)) \rangle$$

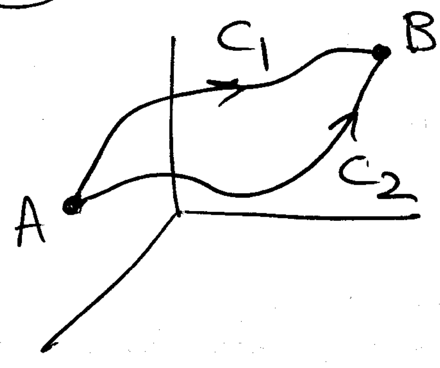
$\nabla\varphi(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$$= \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} = \frac{d}{dt}\varphi(\vec{r}(t)).$$

$$\therefore \int_C \nabla\varphi \cdot d\vec{r} = \int_a^b \frac{d}{dt}\varphi(\vec{r}(t))\,dt = \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$
$$= \varphi(B) - \varphi(A).$$

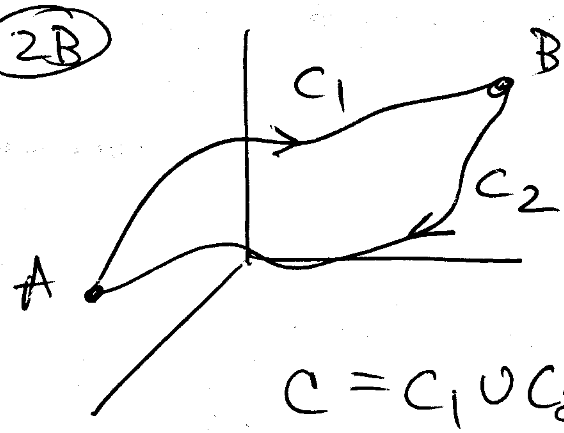
Consequences:

(2A) Path independence:




$$\int_{C_1} \nabla\varphi \cdot d\vec{r} = \int_{C_2} \nabla\varphi \cdot d\vec{r}$$

(2B)




$$\int_C \nabla\varphi \cdot d\vec{r} = 0$$

We would call $C = C_1 \cup C_2$

$C = C_1 \cup C_2$. a closed curve.

integral of a conservative v.f. over a closed curve is zero.

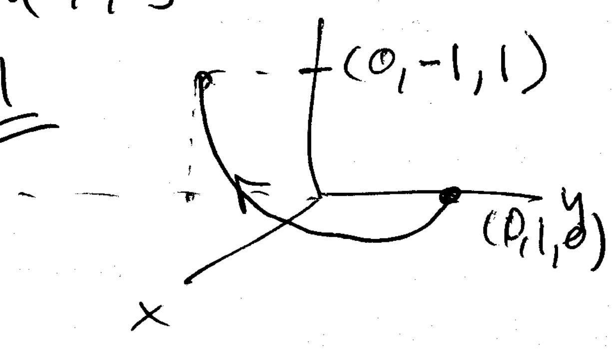
#30) $\int_C \nabla\varphi \cdot d\vec{r}$ $\quad \varphi(x,y,z) = x+y+z$

$C: \vec{r}(t) = \langle \sin t, \cos t, t/\pi \rangle$

$0 \le t \le \pi$

$$\int_C \nabla\varphi \cdot d\vec{r} = \varphi(0,-1,1) - \varphi(0,1,0)$$

$$= 0 - 1 = \underline{\underline{-1}}$$




$\nabla\varphi = \langle 1, 1, 1 \rangle$

$\vec{r}'(t) = \langle \cos t, -\sin t, 1/\pi \rangle$

$$\int_0^{\pi} (\cos t - \sin t + 1/\pi)\, dt$$

$$= \sin t + \cos t + t/\pi \Big|_0^{\pi} = (-1+1) - (1) = -1$$

#38) $\vec{F} = \langle \underset{f}{y-z}, \underset{g}{z-x}, \underset{h}{x-y} \rangle$ $C: \vec{r}(t) = \langle \cos t, \sin t, \cos t \rangle$

$0 \le t \le 2\pi$

$f_y = g_x$ $f_y = 1$ $g_x = -1$ $\vec{F}$ <u>not</u> conservative

$\vec{r}\,'(t) = \langle -\sin t, \cos t, -\sin t \rangle$

$$\int_0^{2\pi} \langle \sin t - \cos t, \cos t - \cos t, \cos t - \sin t \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle \, dt$$

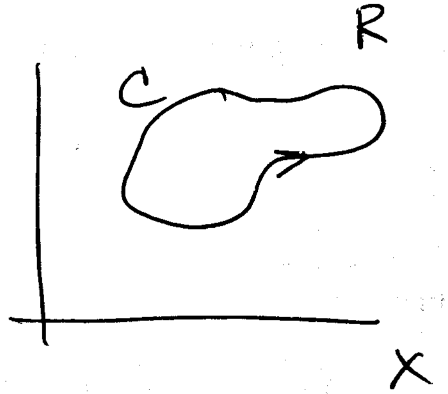
$$= \int_0^{2\pi} -\sin^2 t + \cos t \sin t - \cos t \sin t + \sin^2 t \, dt$$

$$= \int_0^{2\pi} 0 \, dt = 0.$$

Can we say $\vec{F}$ is conservative? No.
We must have $\int_C \vec{F} \cdot d\vec{r} = 0$ for <u>every</u> closed curve $C$, not just one.

# 14.4 Green's Theorem.

Idea: Consider region in the plane.




C is boundary curve of region R.

C is a simple, closed curve


does not cross itself.

$\vec{F}$ - velocity field for some fluid.

(a) $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

circulation around the curve.

(b) $\int_C \vec{F}\cdot\vec{n}\,ds$ flow or flux through C.

Green's Thm says we can evaluate these integrals as double integrals over R.