Change of variables in multiple integrals

$$\iiint_D f(x,y,z)\,dV = \iiint_R \tilde{f}(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

$$x = g(u,v,w)$$
$$y = h(u,v,w)$$
$$z = r(u,v,w)$$

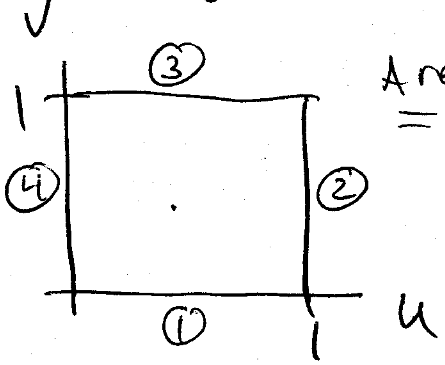
$J(u,v,w)$ = the Jacobian of the transformation given by

e.g. $x = r\cos\theta$, $y = r\sin\theta$ $\qquad J(r,\theta) = r$

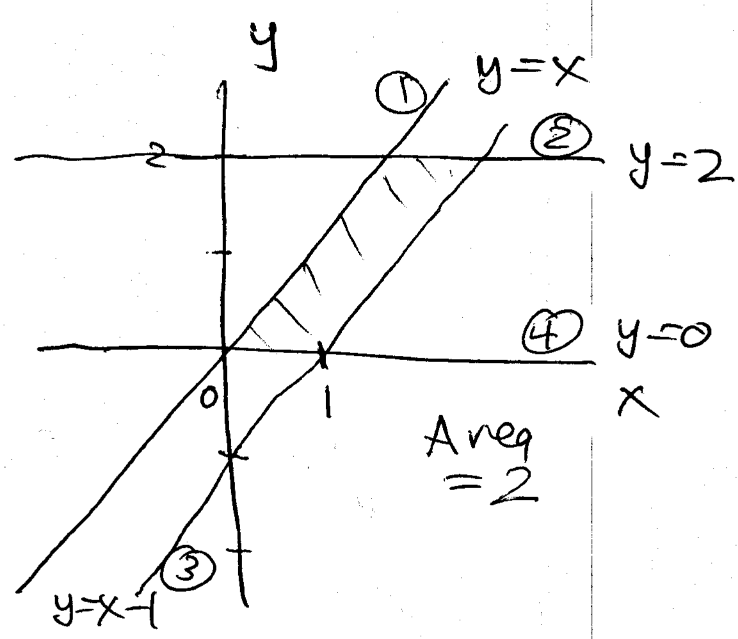
$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta \qquad J(\rho,\varphi,\theta) = \rho^2\sin\varphi.$$
$$z = \rho\cos\varphi$$

eg 8) $x = 2u + v$

$y = 2u$



Area $=1$



① : $v=0$

$$x = 2u,\quad y = 2u \longrightarrow y = x$$

② : $u=1$

$$x = 2+v,\quad y = 2 \longrightarrow y = 2.$$

③ : $v=1$

$$x = 2u+1,\quad y = 2u \longrightarrow y = x-1$$

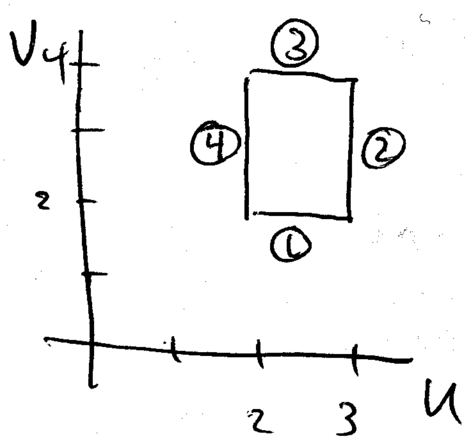
④ : $u=0$

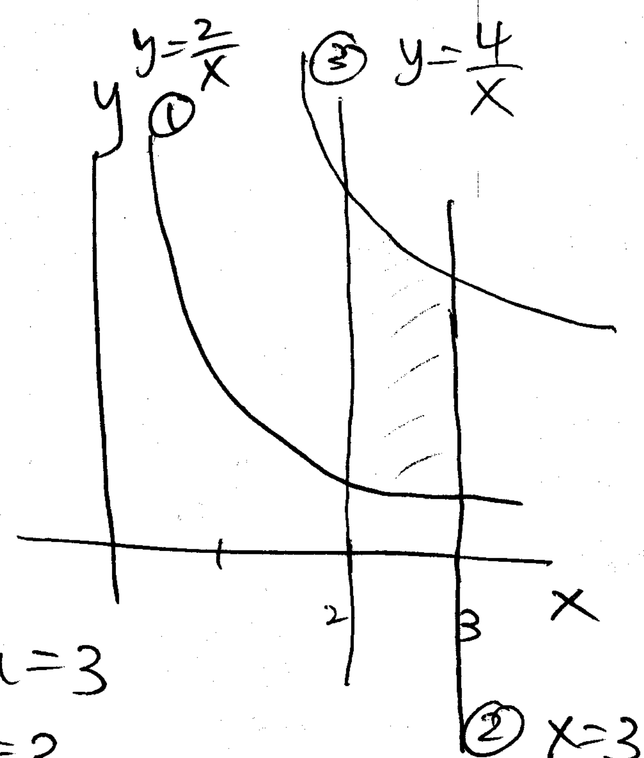
$$x = v,\quad y = 0 \longrightarrow y = 0$$

$$\begin{array}{l} x = 2u+v \longrightarrow \\ y = 2u \longrightarrow \end{array} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = \underbrace{-2}_{J(u,v).}$$

#16) $x = u$

$y = \dfrac{v}{u}$





① : $v=2$

$$x = u,\quad y = \frac{2}{u} \longrightarrow y = \frac{2}{x}$$

② : $u=3$

$$x = 3,\quad y = \frac{v}{3} \longrightarrow x = 3$$

③: $v=4$

$x=4$
$y=\frac{4}{u} \rightarrow y=\frac{4}{x}$

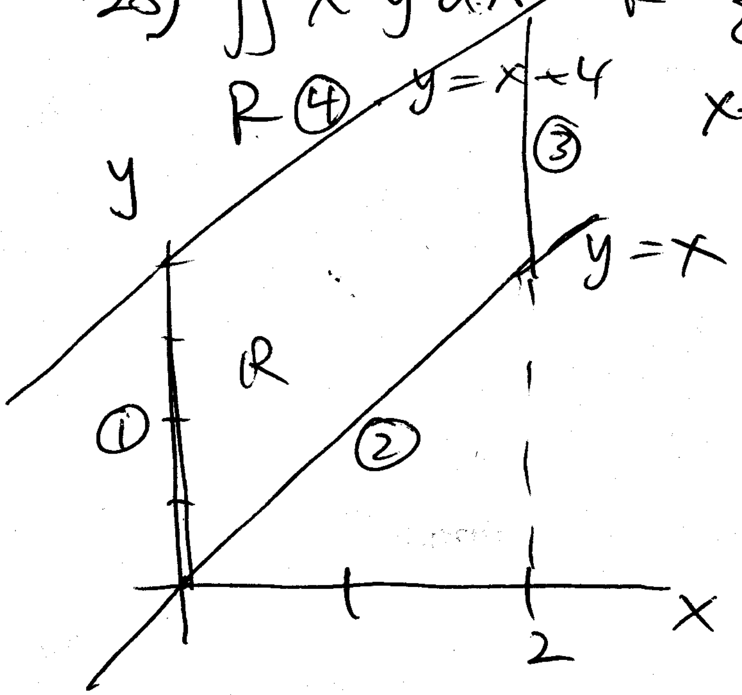
④: $u=2$

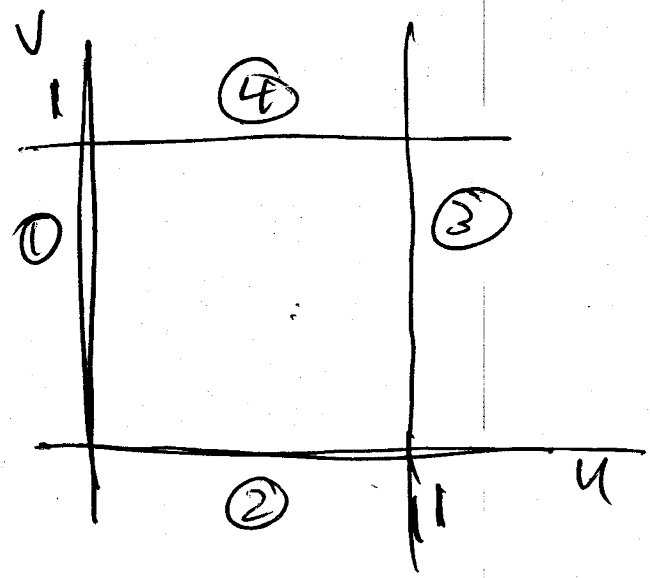
$x=2$
$y=\frac{v}{2} \rightarrow x=2$

$J(u,v)$: $\begin{matrix} x=u \rightarrow \\ y=\frac{v}{u} \rightarrow \end{matrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$

#28) $\iint x^2 y \, dA$ $R=\{(x,y): 0 \le x \le 2, x \le y \le x+4\}$

$x=2u \quad y=4v+2u$





①: $x=0$
$0=2u \rightarrow u=0.$
$y=4v+2u$

②: $y=x$
$4v+2u=2u$
$4v=0$
$v=0.$

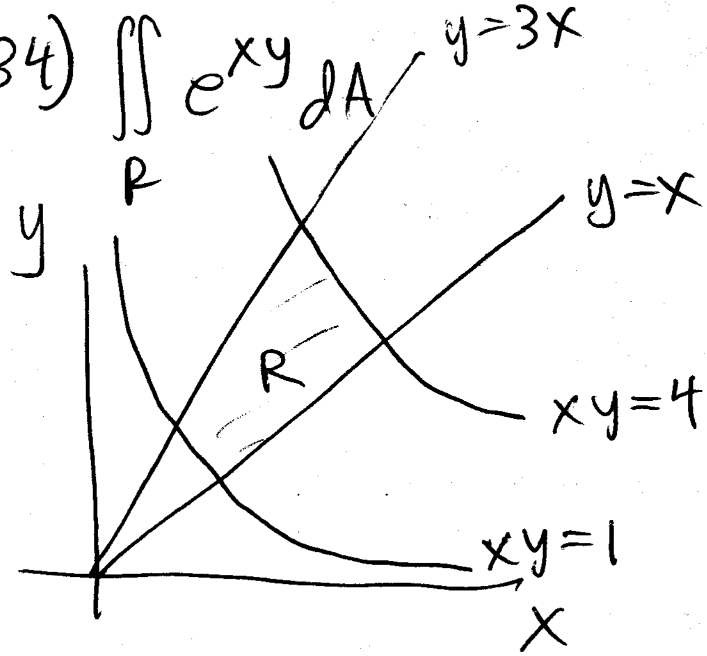
③ $x=2$
$2=2u \rightarrow u=1$
$y=4v+2u$

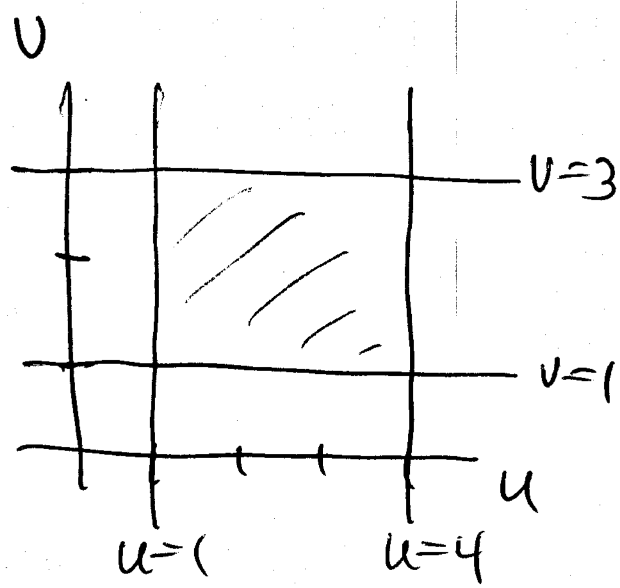
④: $y=x+4$
$4v+2u=2u+4$
$4v=4$
$\underline{v=1}$

$$\begin{matrix} x=2u \longrightarrow \\ y=4v+2u \longrightarrow \end{matrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$$

$$\iint_R x^2 y\,dA = \int_0^1 \int_0^1 (2u)^2(4v+2u)\cdot 8\,dv\,du$$

#34) $\iint_R e^{xy}\,dA$





$xy=1 \rightarrow y=\frac{1}{x}$

$xy=4 \rightarrow y=\frac{4}{x}$

$\frac{y}{x}=1 \quad \frac{y}{x}=3$

$\downarrow \qquad \downarrow$

$y=x \quad y=3x$

Try: $u=xy$

$v=\frac{y}{x}$

$J(u,v)$: $xv=y$

$\hookrightarrow u=x^2v$

$x^2=\frac{u}{v}$

$x=\left(\frac{u}{v}\right)^{1/2}$

$y=v\left(\frac{u}{v}\right)^{1/2}$

$=(uv)^{1/2}$

$$x = \left(\frac{u}{v}\right)^{1/2} \rightarrow \qquad y = (uv)^{1/2} \rightarrow \qquad \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\left(\frac{u}{v}\right)^{-1/2} \cdot \frac{1}{v} & \frac{1}{2}\left(\frac{u}{v}\right)^{-1/2} \frac{-u}{v^2} \\ \frac{1}{2}(uv)^{-1/2} v & \frac{1}{2}(uv)^{-1/2} u \end{vmatrix}$$

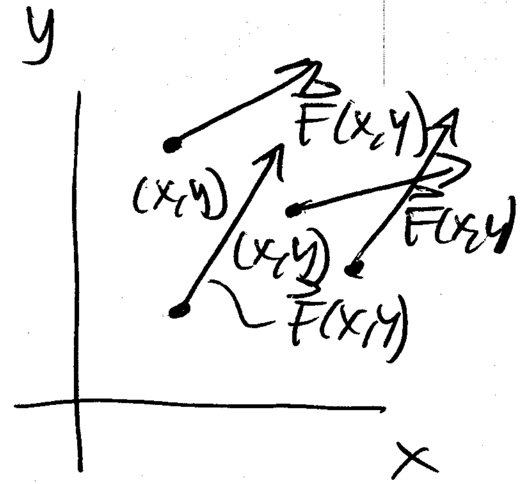
$$= \frac{1}{4}\left(\frac{u}{v}\right)^{-1/2} (uv)^{-1/2} \cdot \frac{u}{v} + \frac{1}{4}\left(\frac{u}{v}\right)^{-1/2} (uv)^{-1/2} \left(+\frac{u}{v}\right)$$

$$= \frac{1}{2}(u^2)^{-1/2} \cdot \frac{u}{v} = \frac{1}{2v} \ //$$

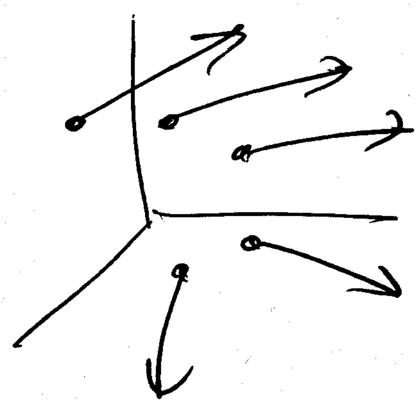
$$\iint_R e^{xy}\, dA = \int_1^3 \int_1^4 e^u \quad \frac{1}{2v}\, du\, dv$$

14.1 Vector Fields.

$$\vec{F}(x,y) = f(x,y)\vec{i} + g(x,y)\vec{j}$$



$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$$



Vector fields usually represent

① velocity fields

② force fields.

Potentials.

$\vec{F} = \nabla \varphi$ ← $\vec{F}$ is a gradient field

φ is a potential function

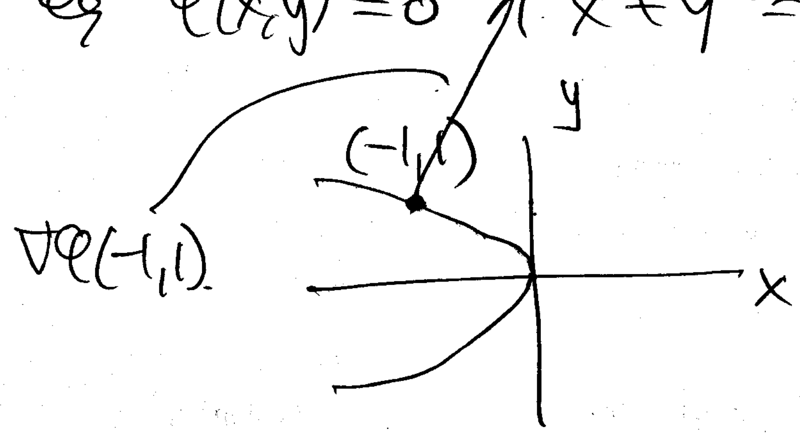
Recall some properties of gradients.

Level curves of $\varphi(x,y)$ are called equipotential curves

<u>Recall</u>: $\nabla\varphi$ is always perpendicular to equipotential curves.

34) $\varphi(x,y) = x + y^2$ $\quad \vec{F} = \nabla\varphi = \vec{i} + 2y\vec{j}$

eg $\varphi(x,y) = 0$ $\quad x + y^2 = 0 \longrightarrow x = -y^2$



$\nabla\varphi(-1,1) = \vec{i} + 2\vec{j}$