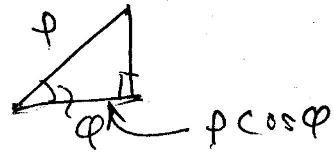
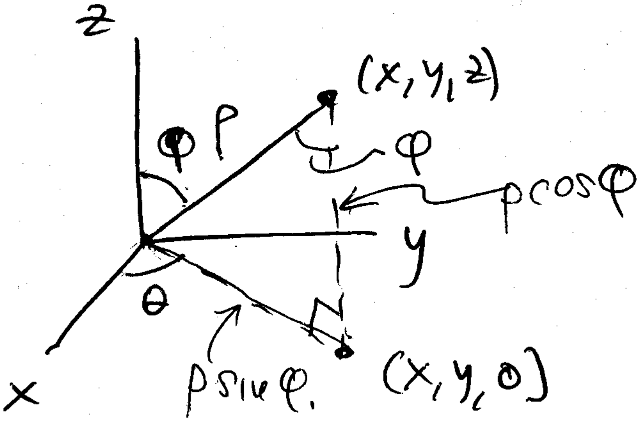
Quiz 11 – 13.3, 13.4

Spherical coordinates





$(x,y,z) \rightarrow (\rho,\varphi,\theta)$

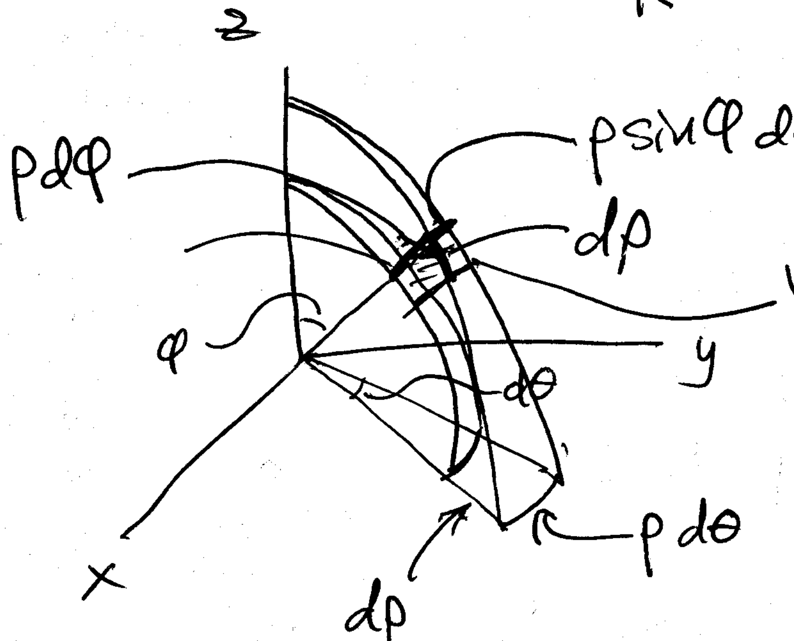
$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$$\iiint_D f(x,y,z)\,dV = \iiint_R \tilde{f}(\rho,\theta,\varphi)\ \underline{\boxed{?}}$$

$\rho^2\sin\varphi\ \underline{d\rho\, d\theta\, d\varphi}$



volume

$= \rho\, d\varphi \cdot \rho\sin\varphi\, d\theta \cdot d\rho$

$= \rho^2\sin\varphi\ \underline{d\rho\, d\theta\, d\varphi}$

#36) $\{(\rho, \varphi, \theta) : \underbrace{\rho = 2\csc\varphi}, \ 0 < \varphi < \pi\}$.

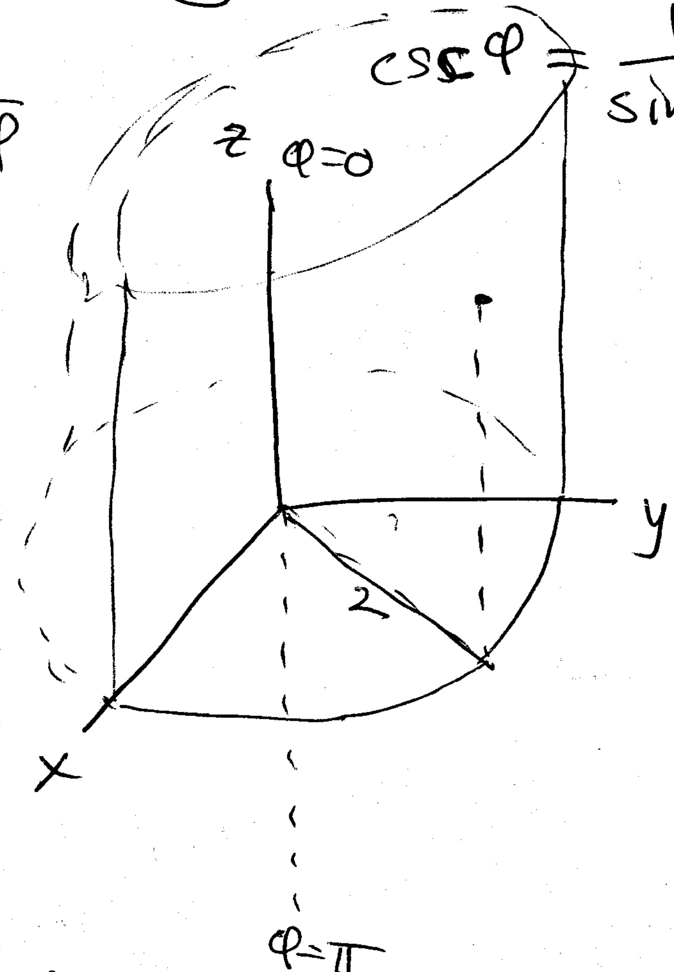
$\rho = 2\csc\varphi = 2\frac{1}{\sin\varphi}$

$\csc\varphi = \frac{1}{\sin\varphi}$

$\underline{\rho\sin\varphi = 2}$

$\underline{0 < \varphi < \pi}$

$\underline{0 \leq \theta < 2\pi}$



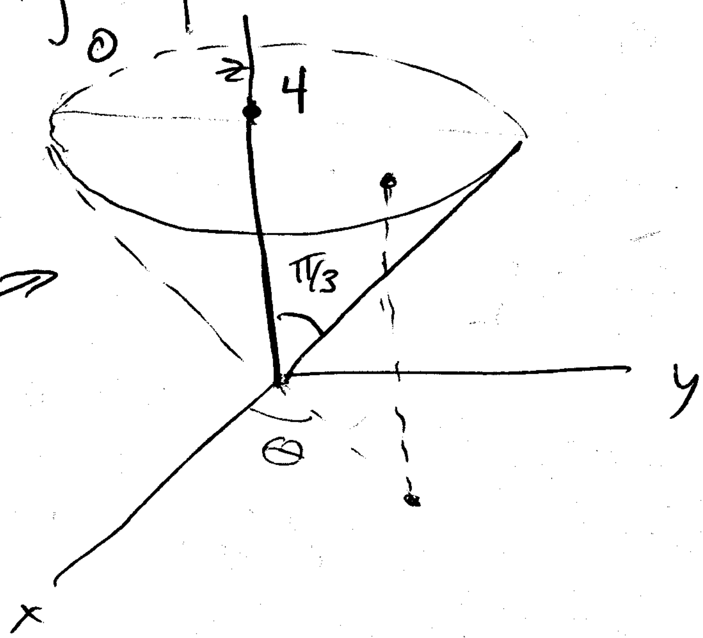
#42) $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\left(\int_0^{4\sec\varphi} \rho^2\sin\varphi\, d\rho\right) d\varphi\, d\theta$

$\rho = 4\sec\varphi = \frac{4}{\cos\varphi}$

$\rho\cos\varphi = 4$

$z = 4$

Volume of this cone



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{3}} \frac{1}{3}\rho^3 \sin\varphi \Big|_0^{4\sec\varphi} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{64}{3} \sec^3\varphi \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{64}{3} \frac{\sin\varphi}{\cos^3\varphi} \, d\varphi \, d\theta$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$\varphi = 0 \longrightarrow u = 1$

$\varphi = \frac{\pi}{3} \longrightarrow u = \frac{1}{2}$

$$= \int_0^{2\pi}\int_{\frac{1}{2}}^{1} \frac{64}{3} u^{-3} \, du \, d\theta . = \cdots$$

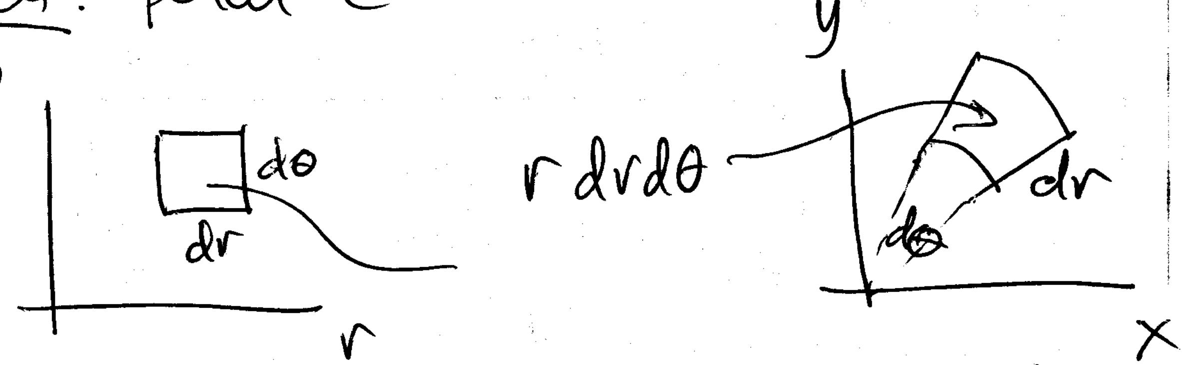
13.7 Change of Variables

General transformations:

$$x = g(u,v) \qquad \longrightarrow \qquad \iint_R f(x,y)\,dA = \iint_S \tilde{f}(u,v)\,du\,dv.$$
$$y = h(u,v)$$

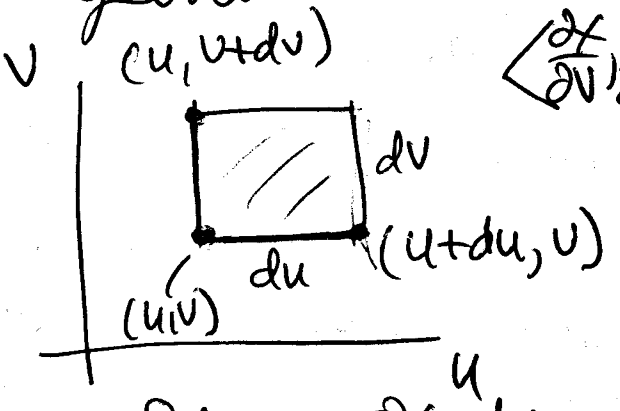
Hardest part: Changing dA.

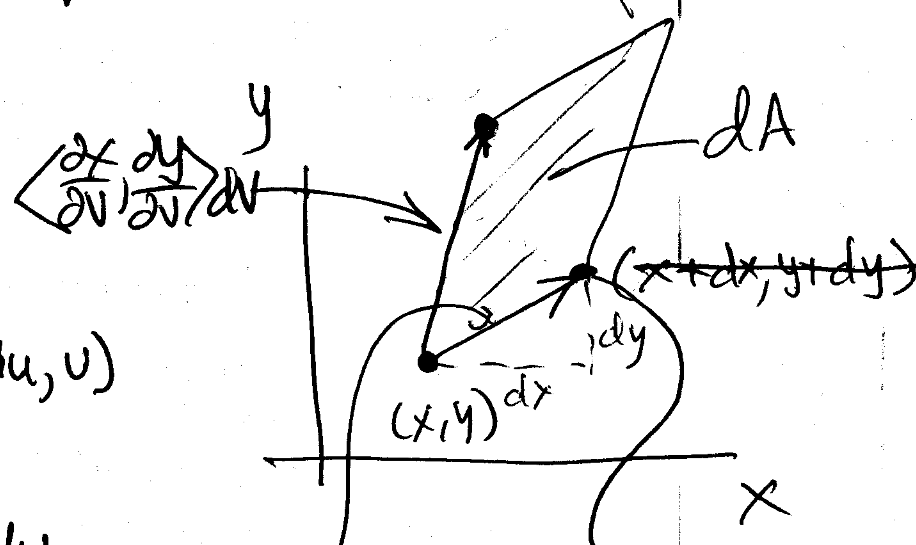
Recall: polar coords.



We found out area of $dr\,d\theta$ "rectangle (polar)" is $r\,dr\,d\theta$.

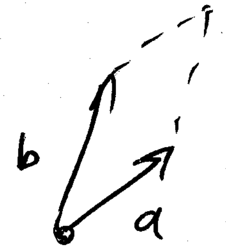
In general.





$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$
$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \rangle du$ $\quad (x + \frac{\partial x}{\partial u}du,\ y + \frac{\partial y}{\partial u}du)$



Area $= |\vec{a} \times \vec{b}|$

$$dA = \left| \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right\rangle du \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right\rangle dv \right|$$

$$= \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{matrix} \right| du\,dv$$

$$= \left| \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right) \vec{k} \right| du\,dv$$

$$= \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right| du\,dv = |J(u,v)|\, du\,dv$$

$\uparrow$ Jacobian

<u>e.g.</u> Polar coords.

$$\begin{matrix} x = r\cos\theta \longrightarrow \\ y = r\sin\theta \longrightarrow \end{matrix} \left| \begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{matrix} \right| = \left| \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \right|$$

$$= \left| r\cos^2\theta - (-r\sin^2\theta) \right| = \left| r(\cos^2\theta + \sin^2\theta) \right|$$
$$= r.$$

Also works in 3-D.

e.g.

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned} \longrightarrow \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$\begin{aligned} &= \sin\varphi\cos\theta\left(\rho^2\sin^2\varphi\cos\theta\right) \\ &\qquad - \rho\cos\varphi\cos\theta\left(-\rho\sin\varphi\cos\varphi\cos\theta\right) \\ &\qquad + \left(-\rho\sin\varphi\sin\theta\right)\left(-\rho\sin^2\varphi\sin\theta - \rho\cos^2\varphi\sin\theta\right) \\ &= \rho^2\sin\varphi\sin^2\varphi\cos^2\theta + \rho^2\cos^2\varphi\cos^2\theta\sin\varphi \\ &\qquad + \rho^2\sin\varphi\sin^2\varphi\sin^2\theta + \rho^2\sin\varphi\cos^2\varphi\sin^2\theta \\ &= \rho^2\sin\varphi\sin^2\varphi + \rho^2\sin\varphi\cos^2\varphi = \rho^2\sin\varphi \end{aligned}$$