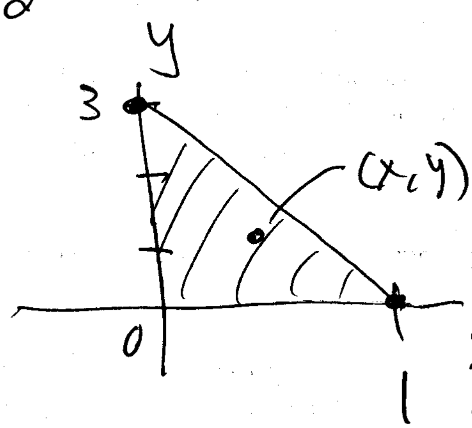
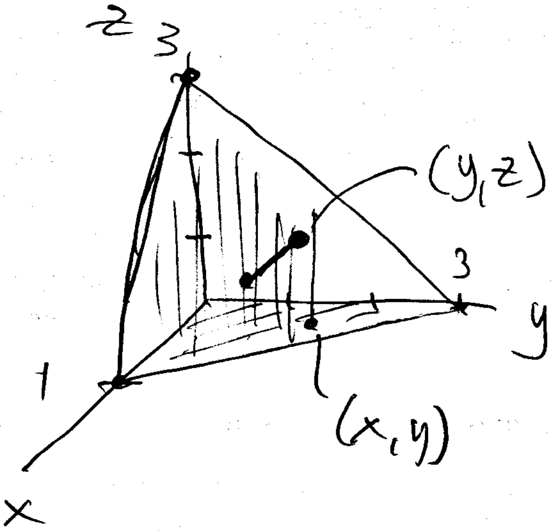
Triple Integrals

$$\iiint_D f(x,y,z)\,dV$$

e.g. $$\int_0^1 \int_0^{3-3x} \left(\int_0^{3-3x-y} 1 \, dz \right) dy \, dx$$

$$\int_0^1 \int_0^{3-3x} \left(\text{——} \right) dy \, dx$$





$y = 3-3x$

$z = 3-3x-y$

~~z~~ $3x+y+z=3$ (plane)

y-z trace: $x=0$

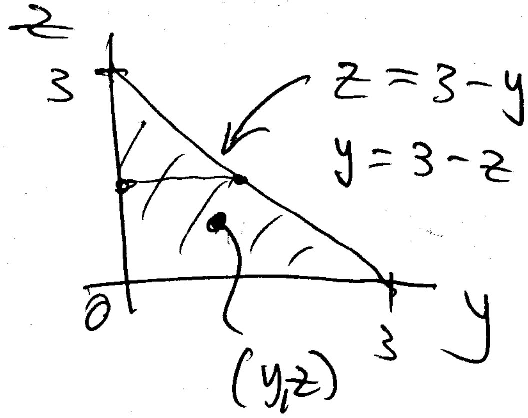
$y+z=3 \quad z=3-y.$

x-z trace: $y=0$

$3x+z=3 \quad z=3-3x$

Suppose we want to change order of integration. Say: we want $dx\,dy\,dz$.

Project onto y-z plane:



$$\int_0^3 \int_0^{3-z} \int_0^{\frac{3-y-z}{3}} dx\,dy\,dz$$

$$3x + y + z = 3$$
$$3x = 3 - y - z$$
$$x = \frac{3-y-z}{3}$$

$$\int_0^1 \int_0^{3-3x} \left(\int_0^{3-3x-y} 1 \, dz \right) dy\,dx$$

$$= \int_0^1 \int_0^{3-3x} \left(z \Big|_0^{3-3x-y} \right) dy\,dx$$

$$= \int_0^1 \left(\int_0^{3-3x} 3-3x-y \, dy \right) dx$$

$$= \int_0^1 \left(3y - 3xy - \frac{1}{2}y^2 \Big|_0^{3-3x} \right) dx$$

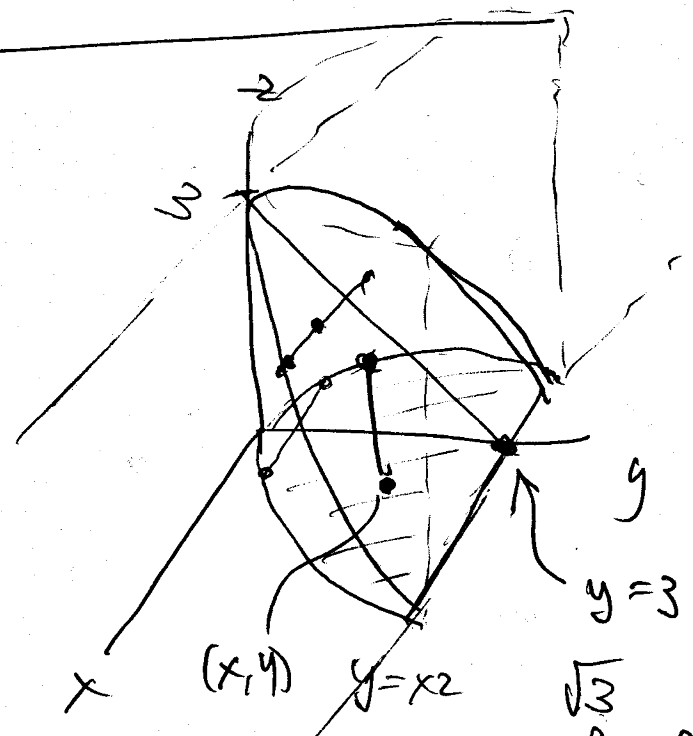
$$= \int_0^1 3(3-3x) - 3x(3-3x) - \frac{1}{2}\underbrace{(3-3x)^2}_{9-18x+9x^2} dx$$

$$= \int_0^1 9 - \cancel{9x} - 9x + 9x^2 - \frac{9}{2} + \cancel{9x} - \frac{9}{2}x^2 \, dx$$

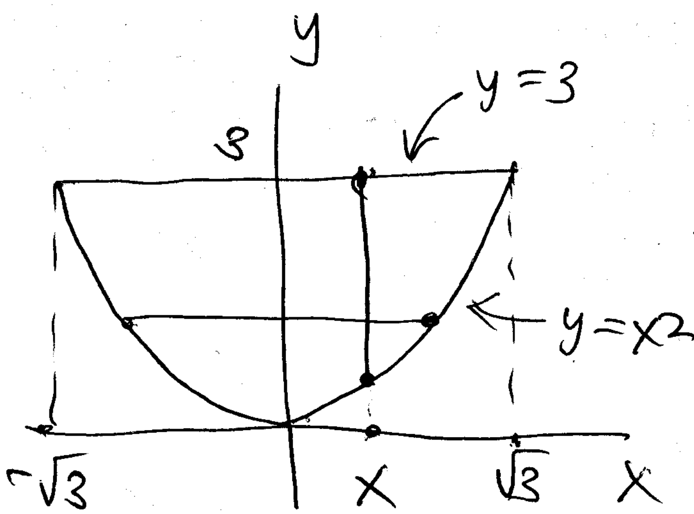
$$= \int_0^1 \frac{9}{2} - 9x + \frac{9}{2}x^2 \, dx$$

$$= \frac{9}{2}x - \frac{9}{2}x^2 + \frac{3}{2}x^3 \Big|_0^1 = \frac{9}{2} - \frac{9}{2} + \frac{3}{2} - 0 = \frac{3}{2}$$

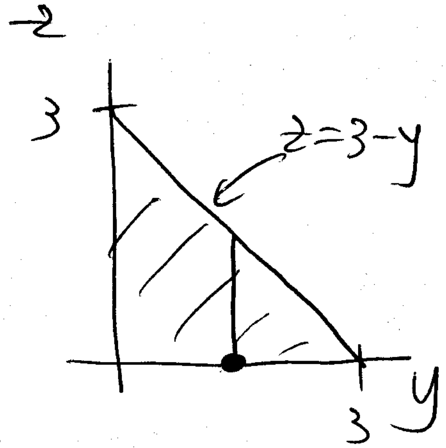
#20)



$z = 3 - y$.



$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_0^{3-y} 1 \, dz \, dy \, dx$$

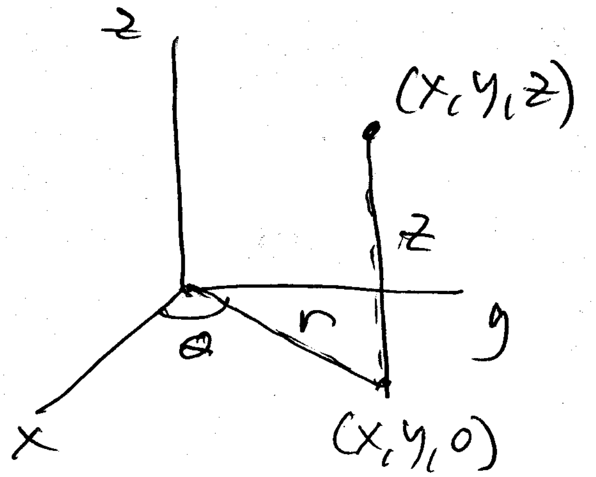


$$\int_0^3 \int_0^{3-y} \int_{-\sqrt{y}}^{\sqrt{y}} 1 \; dx\,dz\,dy$$

13.5 Cylindrical/Spherical Coords.

A. Cylindrical

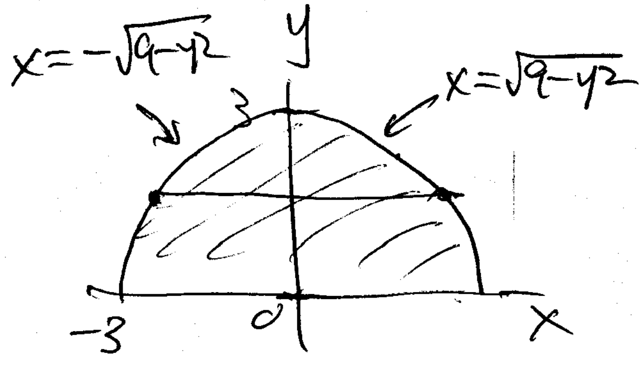
New variables : (r, θ, z)



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

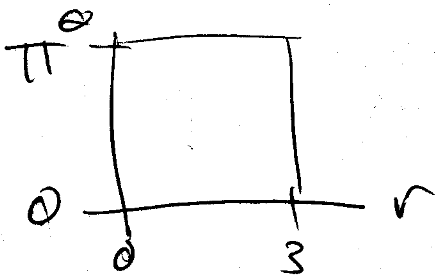
eg "#16)" $$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \left(\int_0^{9-3\sqrt{x^2+y^2}} (\underbrace{x^2+y^2}_{r^2})^{3/2}\, dz \right) dx\,dy.$$

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (\text{——})\, dx\,dy$$



$$x = -\sqrt{9-y^2}$$
$$x^2 = 9-y^2$$
$$x^2+y^2 = 9$$

Polar rectangle:



$$\int_0^{\pi}\int_0^{3} (\text{——})\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{3}\int_0^{9-3r} r^3 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{\pi}\int_0^{3}\left(\int_0^{9-3r} r^4 \, dz\right) dr \, d\theta$$

$$= \int_0^{\pi}\int_0^{3} zr^4 \Big|_0^{9-3r} dr \, d\theta$$

$$= \int_0^{\pi}\int_0^{3} (9-3r)r^4 \, dr \, d\theta$$

$$= \int_0^{\pi}\left(\int_0^{3} 9r^4 - 3r^5 \, dr\right) d\theta$$

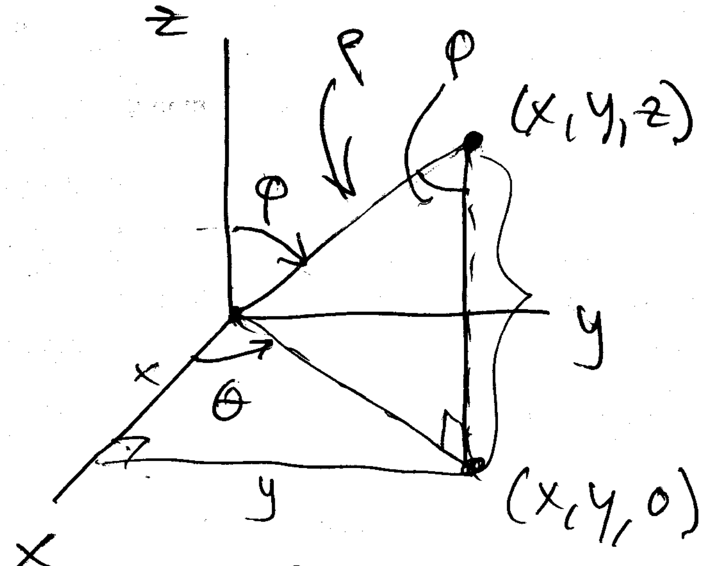
$$= \int_0^{\pi} \frac{9}{5}r^5 - \frac{1}{2}r^6 \Big|_0^{3} d\theta$$

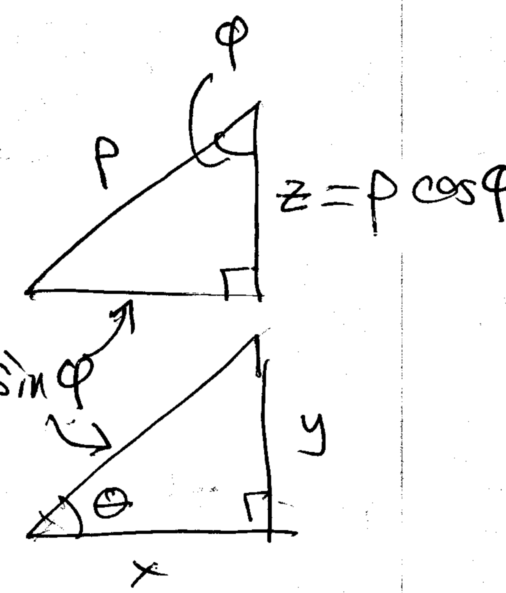
$$= \int_0^{\pi}\left(\left(\frac{9}{5}\right)(243) - \frac{729}{2}\right) d\theta = \pi\left(\frac{9(243)}{5} - \frac{729}{2}\right) \approx \underline{72.9\pi}$$

5

B. Spherical.

(ρ, φ, θ)





$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$$\iiint_D f(x,y,z)\, dV = \int\int\int f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \boxed{?}$$