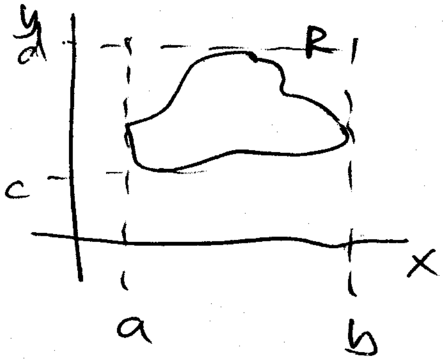
Double integrals



Can evaluate as iterated integral in at least one way, maybe two.

13.3 Polar coordinates

Idea: e.g. $\int_0^{\pi} 2x\cos(x^2)\,dx$

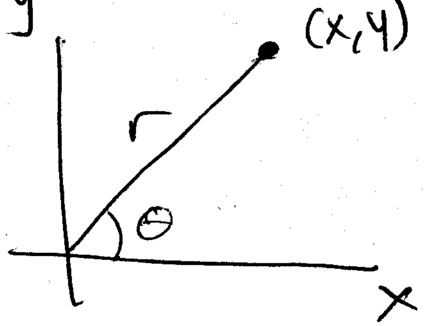
change variables: $u=x^2$ $du=2x\,dx$

$x=0 \to u=0$
$x=\pi \to u=\pi^2$

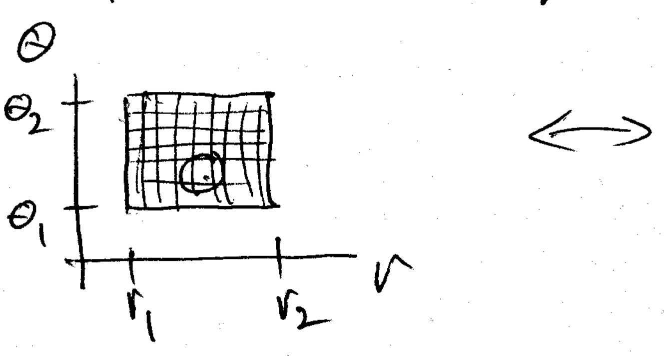
$$= \int_0^{\pi^2} \cos(u)\,du$$

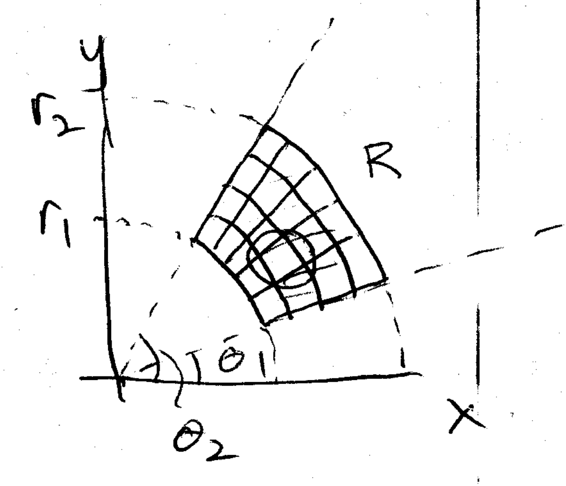
How can we change variables in 2-dim?

Polar coordinates: $x=r\cos\theta$
$y=r\sin\theta$
$r^2=x^2+y^2$

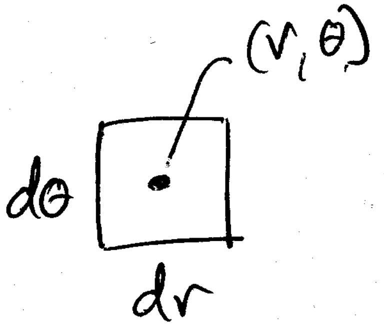


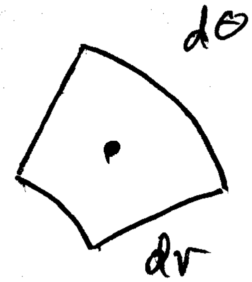
A. Polar rectangles.

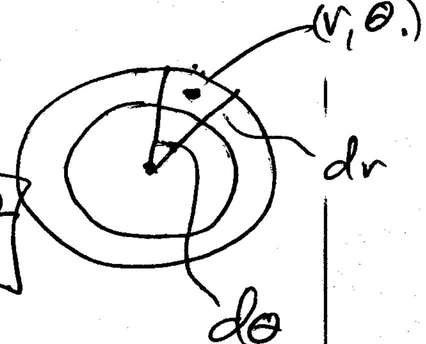




$$? \Longleftarrow \iint_R f(x,y)\, dA$$







Area of ring:

$$= \left[\pi\left(r+\frac{dr}{2}\right)^2 - \pi\left(r-\frac{dr}{2}\right)^2\right] \times \left(\frac{d\theta}{2\pi}\right)$$

$$= \cancel{\pi}\left[\left(\cancel{r^2} + r\,dr + \cancel{\frac{dr^2}{4}}\right) - \left(\cancel{r^2} - r\,dr + \cancel{\frac{dr^2}{4}}\right)\right]\left(\frac{d\theta}{2\cancel{\pi}}\right)$$

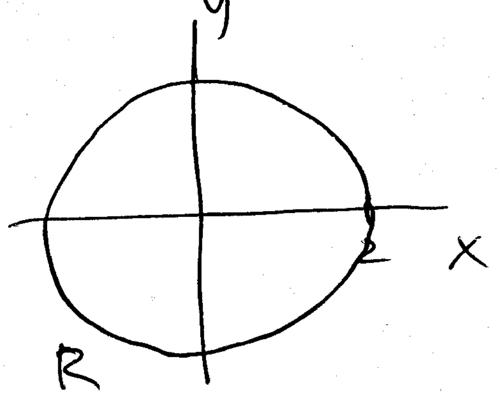
$$= \underline{r\,dr\,d\theta}$$

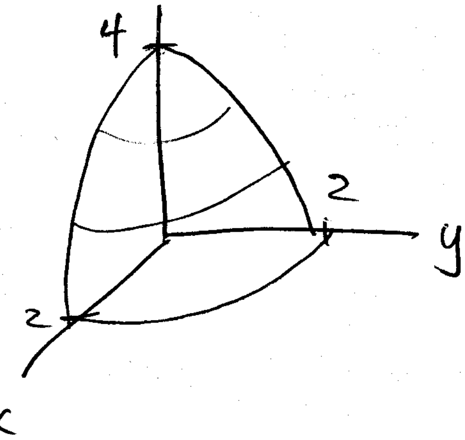
So....

$$\iint_R f(x,y)\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

eg #12) $z = 4 - x^2 - y^2$

$R = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}.$



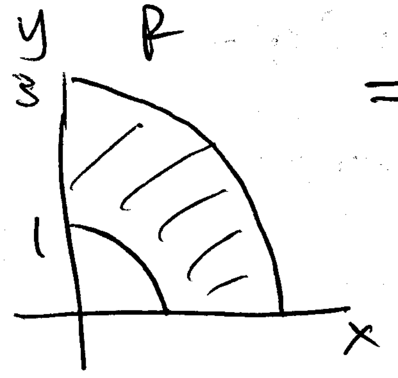


$$\iint_R (4-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^{2} (4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{2} 4r - r^3\,dr\right)d\theta = \int_0^{2\pi}\left(2r^2 - \frac{1}{4}r^4\Big|_0^2\right)d\theta$$

$$= \int_0^{2\pi} (8-4) - 0\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

#20) $\iint 2xy\,dA \quad R = \{(r,\theta) : 1 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2}\}.$



$$= \int_0^{\pi/2}\int_1^{3} (2\cdot r\cos\theta \cdot r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^{3} 2r^3 \sin\theta\cos\theta\,dr\,d\theta$$

$$= \int_{\alpha}^{\pi/2} \left(\int_1^3 r^3 \sin(2\theta)\, dr \right) d\theta$$

$$= \int_{\alpha}^{\pi/2} \frac{1}{4} r^4 \sin 2\theta \Big|_1^3 \, d\theta$$

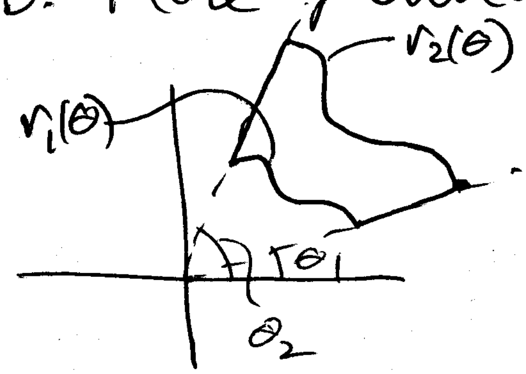
$$= \int_{\alpha}^{\pi/2} \left(\frac{81}{4} - \frac{1}{4} \right) \sin 2\theta \, d\theta$$

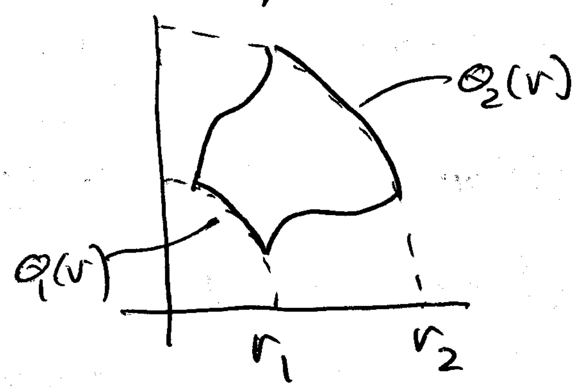
$$= 20 \int_{\alpha}^{\pi/2} \sin 2\theta \, d\theta$$

$$= 20 \cdot \left(-\frac{1}{2} \cos 2\theta \Big|_0^{\pi/2} \right)$$

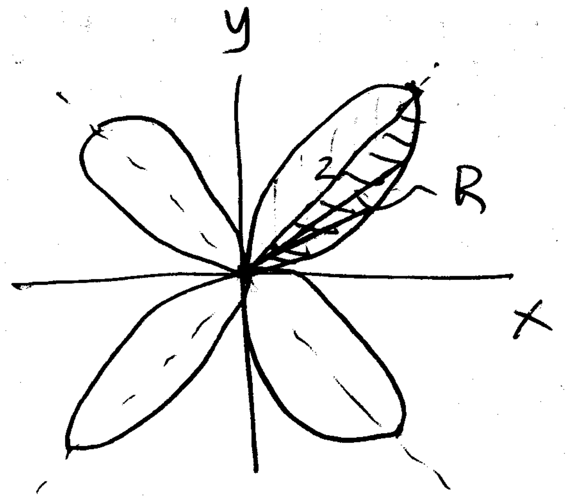
$$= 20 \left(\frac{1}{2} + \frac{1}{2} \right) = 20.$$

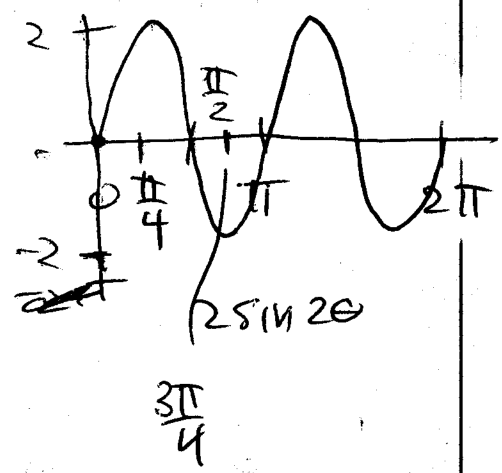
B. More general polar regions.





eg #30) $r = 2\sin 2\theta$





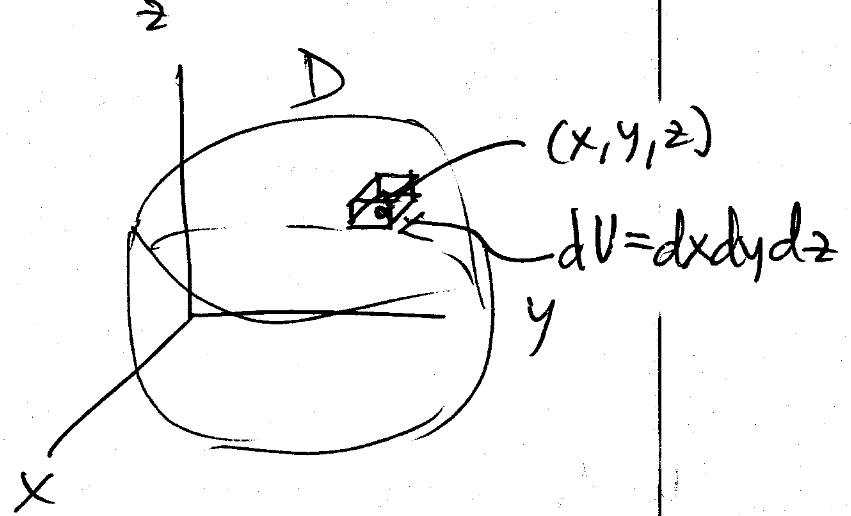
$$A = 2\iint_R dA = 2\int_0^{\pi/4}\int_0^{2\sin 2\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4}\left(\frac{1}{2}r^2\Big|_0^{2\sin 2\theta}\right)d\theta$$

$$= 2\int_0^{\pi/4} 2\sin^2(2\theta)\,d\theta = \cdots$$

13.4 Triple Integrals

$$\iiint_D F(x,y,z)\,dV$$



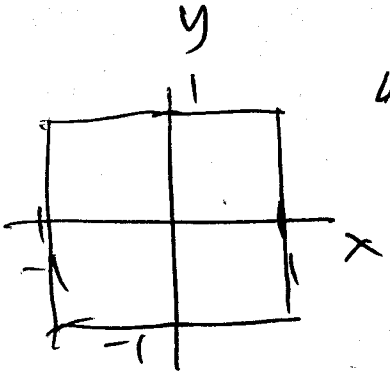
Interpretation: $F(x,y,z) =$ density of material at (x,y,z)

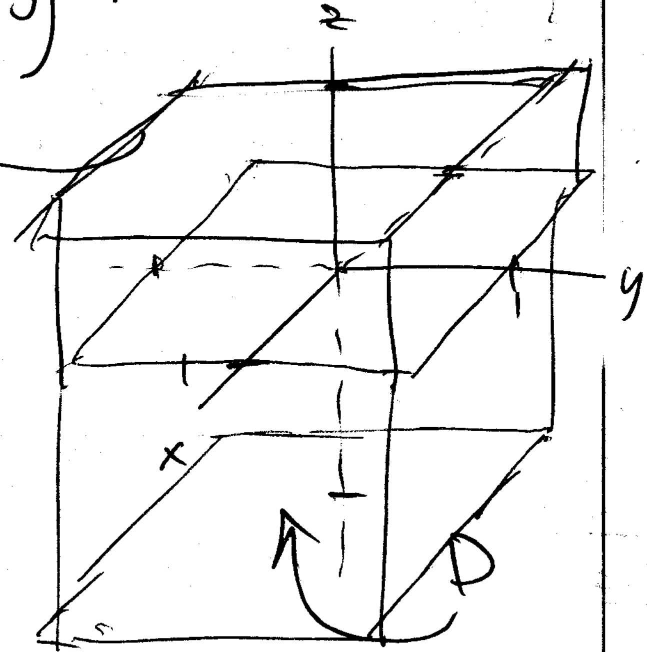
$F(x,y,z)\,dV =$ mass of small cube at (x,y,z)

$\iiint_D F(x,y,z)\,dV =$ total mass.

But: $\iiint_D 1\,dV =$ total volume of D.

e.g. $\displaystyle\int_{-1}^{1}\int_{-1}^{1}\left(\int_{-1}^{1}(x+y+z)\,dy\right)dx\,dz$





$$\int_{-1}^{1}\int_{-1}^{1}\left(xy+\tfrac{1}{2}y^2+zy\Big|_{-1}^{1}\right)dx\,dz$$

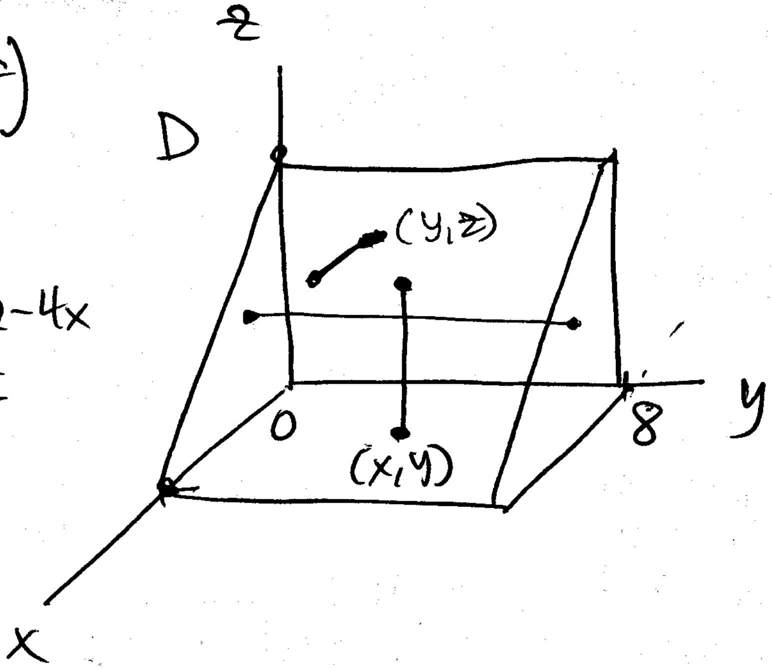
$$=\int_{-1}^{1}\int_{-1}^{1}\left(x+\cancel{\tfrac{1}{2}}+z\right)-\left(-x+\cancel{\tfrac{1}{2}}-z\right)dx\,dz$$

$$=\int_{-1}^{1}\left(\int_{-1}^{1}(2x+2z)\,dx\right)dz=\int_{-1}^{1}\left(x^2+2zx\right)\Big|_{-1}^{1}\,dz$$

$$=\int_{-1}^{1}\left(\cancel{1}+2z\right)-\left(\cancel{1}-2z\right)dz$$

$$=\int_{-1}^{1}4z\,dz=2z^2\Big|_{-1}^{1}=2-2=0$$

#18)

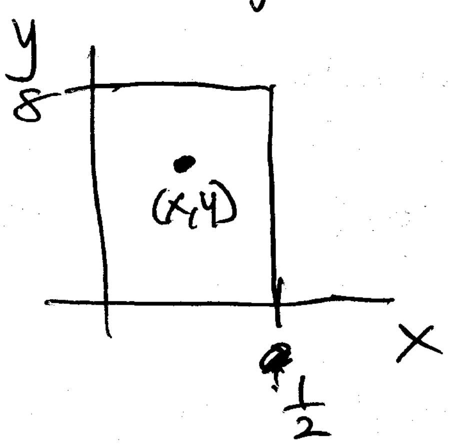


$z = 2-4x$
$y = 8$

$0 = 2-4x$
$x = \frac{1}{2}$

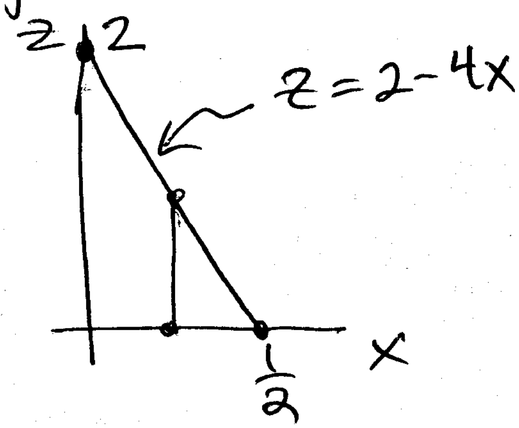
Set up $\iiint_D f(x,y,z)\,dV$.

Idea: Project D onto x-y plane



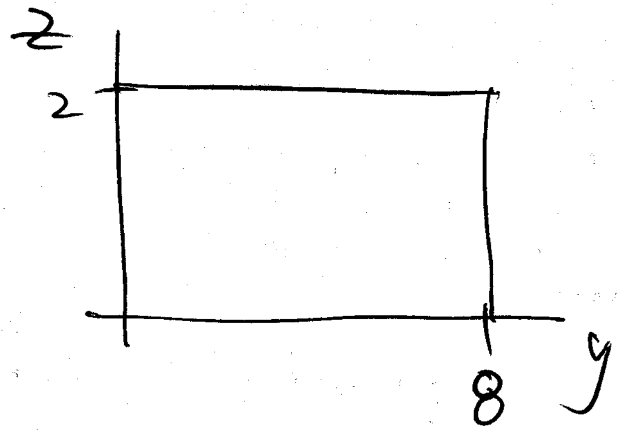
$$\int_0^8 \int_0^{1/2} \int_0^{2-4x} f(x,y,z)\,dz\,dx\,dy$$

Project onto x-z plane



$$\int_0^{1/2} \int_0^{2-4x} \int_0^8 f(x,y,z)\,dy\,dz\,dx$$

Project onto z-y plane



$$\int_0^8 \int_0^2 \int_0^{\frac{2-z}{4}} f(x,y,z)\,dx\,dz\,dy$$

$$z = 2 - 4x$$

$$x = \frac{2-z}{4}$$