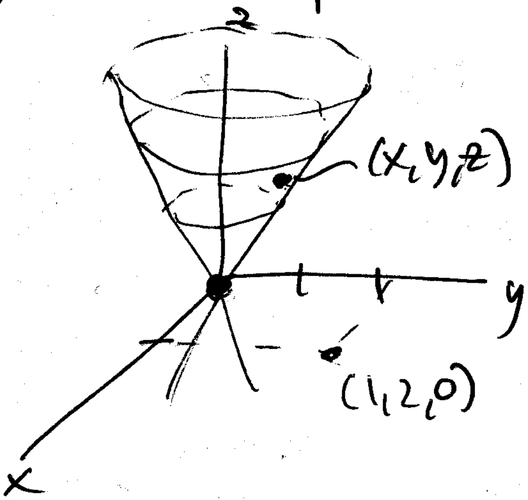
Exam 2 · Tuesday 4/3 11.9 – 12.9
Also Quiz 9 on same day.

Lagrange multipliers.

eg #24)



$z^2 = x^2 + y^2$

Find point on cone closest to $(1,2,0)$.

Minimize:

$d = \left((x-1)^2 + (y-2)^2 + z^2\right)^{1/2}$

subject to:

$\underbrace{z^2 - x^2 - y^2}_{g(x,y)} = 0.$

Simplifying: minimizing d is same as minimizing d^2. So sufficient to minimize

$f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2.$

$\nabla f = \langle 2(x-1), 2(y-2), 2z \rangle = 2\langle x-1, y-2, z \rangle$

$\nabla g = \langle -2x, -2y, 2z \rangle = 2\langle -x, -y, z \rangle$ $\boxed{\nabla f = \lambda \nabla g}$

$x - 1 = -\lambda x \longrightarrow yx - y = -\lambda xy$

$y - 2 = -\lambda y \longrightarrow xy - 2x = -\lambda xy$

$z = \lambda z$

$\cancel{yx} - y = \cancel{xy} - 2x$

$z^2 = x^2 + y^2 \longleftarrow$ $y = 2x$

①

$z^2 = x^2 + (2x)^2 = 5x^2$ $\quad z^2 = 5x^2$

$z = \lambda z \longrightarrow$ $\underline{z = 0}$ $\quad \underline{\lambda = 1}$

$\downarrow$

$\underline{x = 0}$

$\underline{y = 0}$

$\underline{(0,0,0)}$

$x - 1 = -x$

$2x = 1$

$x = \frac{1}{2}$ $\quad y = 1$ $\quad \cancel{z = \pm\sqrt{5}}$

$z = \pm \frac{\sqrt{5}}{2}$

Check:

$\underline{\left(\frac{1}{2}, 1, \frac{\sqrt{5}}{2}\right)}$ $\quad \underline{\left(\frac{1}{2}, 1, \frac{-\sqrt{5}}{2}\right)}$

$f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$

$f(0,0,0) = 5$, $\quad f\left(\frac{1}{2}, 1, \frac{\sqrt{5}}{2}\right) = f\left(\frac{1}{2}, 1, \frac{-\sqrt{5}}{2}\right)$

$= \frac{1}{4} + 1 + \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$

So distance is minimized at $\left(\frac{1}{2}, 1, \pm\frac{\sqrt{5}}{2}\right)$.

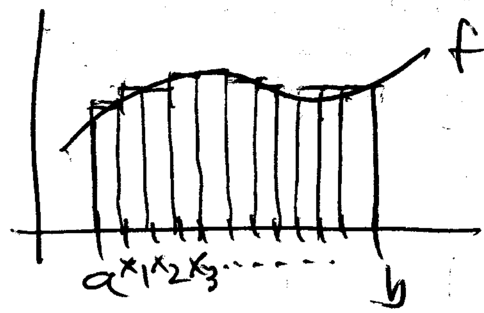
$xz = \lambda z x$

$\cancel{xz - x = \lambda x^2}$

$xz - z = -\lambda x z$

13.1 Double Integrals over Rectangles.

Rcl: $\int_a^b f(x)\,dx$ = "area under graph $y=f(x)$"



① Partition $[a,b]$ into small subintervals

② Form Riemann sum $\sum \underbrace{f(x_i^*)}_{\text{height of rect. over ith subinterval}} \underbrace{\Delta x_i}_{\text{length of ith subinterval}}$

Approximates area under curve.

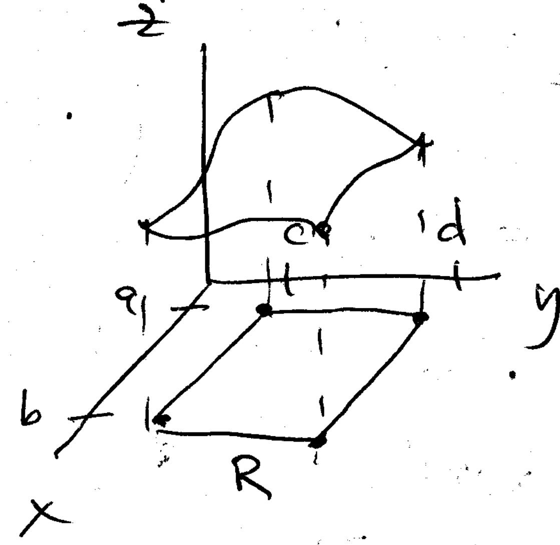
③ Take limit as size of partition $\to 0$

$$\sum f(x_i^*)\Delta x_i \longrightarrow \int_a^b f(x)\,dx$$

④ $\int_a^b f(x)\,dx = F(b)-F(a)$ where $F'(x)=f(x)$.
(F.T.C.)

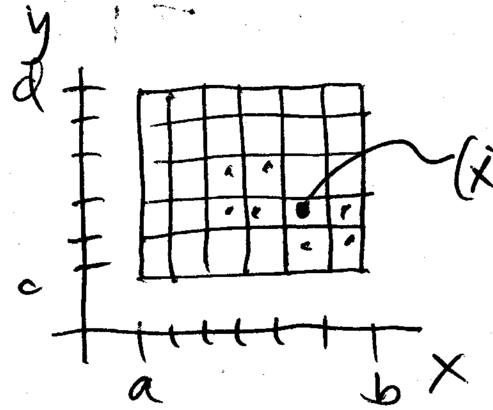
In 2-dim $z = f(x,y)$. Integrate over a rectangle $[a,b] \times [c,d]$.

Compute



$$\iint_R f(x,y)\,dA = \text{"volume of solid under graph of } z = f(x,y)\text{"}$$

<u>Idea</u>: ① Partition $[a,b] \times [c,d]$ by partitioning $[a,b]$ and $[c,d]$ separately



② Form Riemann sum

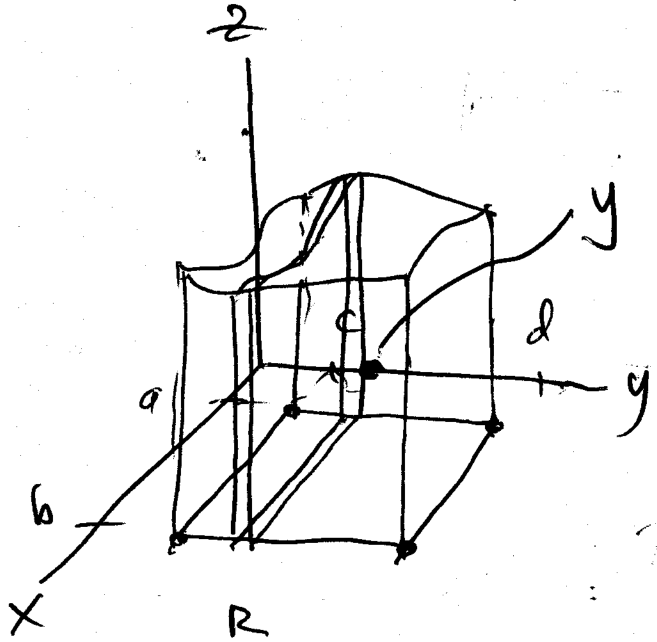
$$\sum \sum f(x_i^*, y_i^*) \underbrace{\Delta A_i}_{\text{area of } i\text{th small rectangle.}}$$

③ Take limit as partition $\longrightarrow 0$.

$$\sum \sum f(x_i^*, y_i^*) \Delta A_i \to \iint_R f(x,y)\,dA.$$

④ FTC is complicated but we can compute these as <u>iterated integrals</u>.

Iterated Integrals.

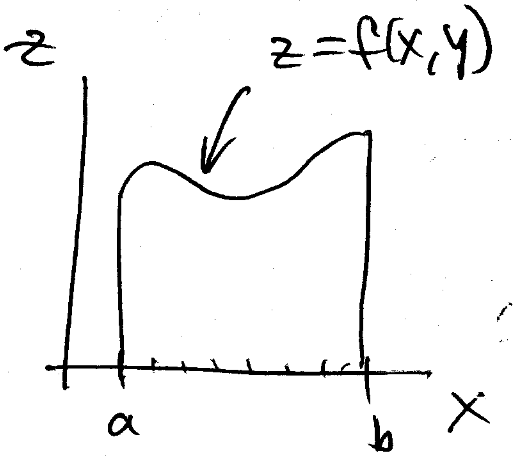


Idea:

Fix $y \in [c,d]$

Area of slice at $y =$

$$\int_a^b f(x,y)\,dx$$



$$\left(\int_a^b f(x,y)\,dx\right) dy = \text{volume of fattened slice at } y.$$

$$\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \text{volume under surface.}$$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

e.g. $\displaystyle\int_0^3 \left[\int_{-2}^0 (x^2y - 2xy)\,dy\right] dx$

$$= \int_0^3 \left[\left(\tfrac{1}{2}x^2y^2 - xy^2\right)\Big|_{-2=y}^{0=y}\right] dx$$

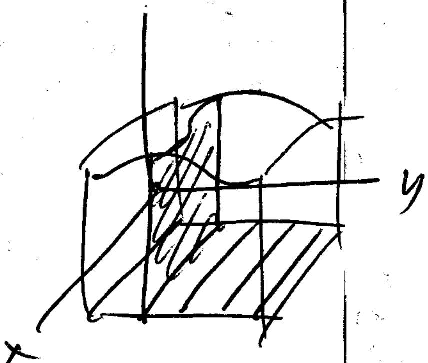
$$= \int_0^3 \left[0 - \left(\tfrac{1}{2}x^2(-2)^2 - x(-2)^2\right)\right] dx$$

this should be a function of x alone.

$$= \int_0^3 (-2x^2 + 4x)\, dx$$

$$= -\frac{2}{3}x^3 + 2x^2 \Big|_0^3 = -18 + 18 - 0 = 0.$$

$$\int_{-2}^{0} \left[\int_0^3 (x^2y - 2xy)\, dx\right] dy$$

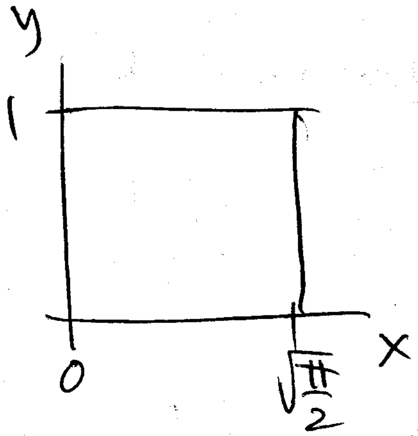


$$= \int_{-2}^{0} \left[\left(\tfrac{1}{3}x^3y - x^2y\right)\Big|_{0=x}^{3=x}\right] dy$$

$$= \int_{-2}^{0} 9y - 9y - 0\, dy = \int_{-2}^{0} 0\, dy = 0.$$

#16) $\iint_R xy\sin(x^2)\,dA$

$R = \{(x,y) : 0 \le x \le \sqrt{\tfrac{\pi}{2}},\ 0 \le y \le 1\}.$



$$\int_0^1 \left[\int_0^{\sqrt{\frac{\pi}{2}}} xy\sin(x^2)\,dx \right] dy$$

$$= \int_0^1 -\tfrac{1}{2}y\cos(x^2)\Big|_0^{\sqrt{\frac{\pi}{2}}}\,dy$$

$$= \int_0^1 0 - \left(-\tfrac{1}{2}y(1)\right) dy$$

$$= \int_0^1 \tfrac{1}{2}y\,dy$$

$$= \tfrac{1}{4}y^2\Big|_0^1 = \tfrac{1}{4}.$$

$\int xy\sin(x^2)\,dx$

$= y\cdot\frac{1}{2}\underbrace{\int 2x\sin(x^2)\,dx}$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}y\int \sin(u)\,du$

$= -\frac{1}{2}y\cos(u)$

$= -\frac{1}{2}y\cos(x^2)$