Max/Min problems.

Idea: $z = f(x,y)$ want to find all local extrema of f.

① Find critical points: $\nabla f(x,y) = \vec{0}$ or $\nabla f(x,y)$ is not defined.

If $\nabla f(x,y) = \vec{0}$ then tangent plane has the form $z = \text{constant}$.

i.e. Spse $\nabla f(a,b) = \vec{0}$

T.P: $\vec{n} = f_x(a,b)\vec{i} + f_y(a,b)\vec{j} - \vec{k} = -\vec{k}$

$P_0 = (a, b, f(a,b))$

$(0)(x-a) + (0)(y-b) + (-1)(z - f(a,b)) = 0$

$z = f(a,b)$ //

② 2nd Derivative Test.

Def: The discriminant or Hessian of $f(x,y)$ is defined to be

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Test: If $\nabla f(a,b) = \vec{0}$:

(i) $f_{xx} < 0$ (or f_{yy}) and $D(a,b) > 0$: local max

(ii) $f_{xx} > 0$ (or f_{yy}.) and $D(a,b) > 0$: local min

(iii) $D(a,b) < 0$: saddle point

(iv) $D(a,b) = 0$: inconclusive

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eg $f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

critical point : $(15, -8)$

$f_x = 2x + 3y - 6$ $\quad$ $f_y = 3x + 6y + 3$

$f_{xx} = 2$ $\quad$ $f_{yy} = 6$

$f_{xy} = 3 = f_{yx}$ $\quad$ $D(x,y) = (2)(6) - (3)^2 = 3 > 0$.

Since $f_{xx} > 0$: local min.

Note: $D(15,-8) = 3 > 0$.

e.g. $f(x,y) = 8x^3 + y^3 + 6xy$

$f_x = 24x^2 + 6y \qquad f_y = 3y^2 + 6x$

$24x^2 + 6y = 0 \rightarrow 4x^2 + y = 0$

$3y^2 + 6x = 0 \rightarrow y^2 + 2x = 0$

$y = -4x^2$

$(-4x^2)^2 + 2x = 0$

$16x^4 + 2x = 0$

$2x(8x^3 + 1) = 0 \rightarrow \underline{x = 0} \quad \underline{x = \frac{-1}{2}}$

Crit pts: $(0,0)$ $(-\frac{1}{2}, -1)$. $\quad \underline{y = 0} \quad \underline{y = -1}$

$f_{xx} = 48x \qquad f_{yy} = 6y \qquad f_{xy} = 6$

$D(x,y) = 288xy - 36$

① : $(0,0)$

$f_{xx}(0,0) = 0$

$D(0,0) = -36 < 0$

<u>saddle point</u>

② : $(-\frac{1}{2}, -1)$

$f_{xx}(-\frac{1}{2}, -1) = -24 < 0$

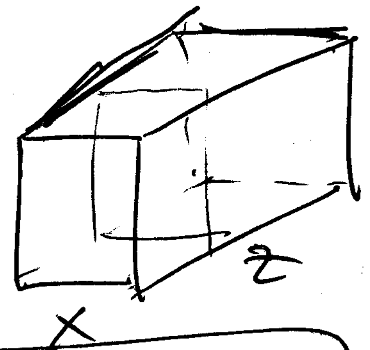
$D(x,y) = 144 - 36 > 0$

<u>local max</u>

12.9 Lagrange Multipliers

Problem: Maximize/minimize an objective function $f(x,y,z)$ subject to a constraint $g(x,y,z)=0$.

#17)



Maximize volume

$V = xyz$ objective

subject to:

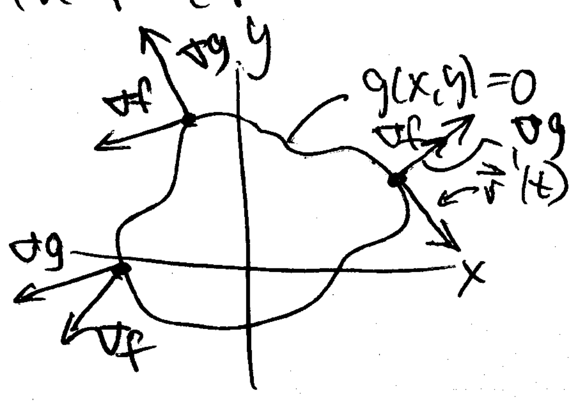
$z + 2x + 2y = 108$ constraint

$$\boxed{\begin{aligned} f(x,y,z) &= xyz \\ g(x,y,z) &= z+2x+2y-108 \end{aligned}}$$

Idea: Consider a 2-variable problem:

Max/min $f(x,y)$ subject to $g(x,y)=0$.

① $g(x,y)=0$ defines a curve in plane. In fact it is a level curve of $z=g(x,y)$.



Suppose curve can be written as graph of a vector function $\vec{r}(t)$.

$\vec{r}(t) = \langle x(t), y(t) \rangle$.

② So we are maximizing (or minimizing) the function $f(x(t), y(t))$. So we must have $\frac{d}{dt}\left(f(x(t), y(t))\right) = 0$.

③ $\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

$= \nabla f \cdot \vec{r}\,'(t) = 0$

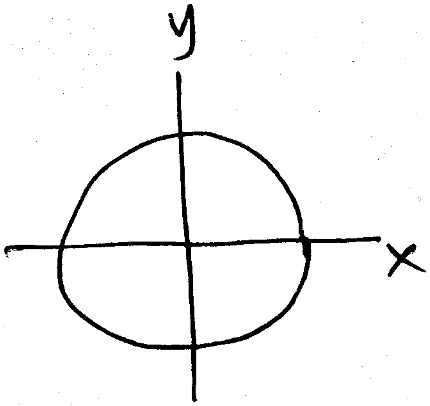
This means: $\boxed{\nabla f \text{ is normal to the curve } g(x,y)=0.}$

④ Since $g(x,y)=0$ is a level curve of $z=g(x,y)$, $\boxed{\nabla g(x,y) \text{ is also normal to curve } g(x,y)=0}$

⑤ At the point where f is max/min, $\boxed{\nabla g \text{ is parallel to } \nabla f.}$ That is for some λ, $\boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$

λ – Lagrange multiplier.

#6) $f(x,y) = xy^2 \qquad x^2+y^2=1$



$g(x,y) = x^2+y^2-1$

$\nabla f = \langle y^2, 2xy \rangle$

$\nabla g = \langle 2x, 2y \rangle$

Solve $\nabla f = \lambda \nabla g$.

$y^2 = 2\lambda x \longrightarrow y^3 = 2\lambda xy$

$2xy = 2\lambda y \longrightarrow 2x^2y = 2\lambda xy \longrightarrow y^3 = 2x^2y$

$x^2+y^2=1 \longrightarrow x^2+2x^2=1$

$3x^2=1$

$x = \pm \frac{1}{\sqrt{3}}$

$y^3 = 2x^2y$

$y^3 - 2x^2y = 0$

$y(y^2-2x^2)=0$

$\underline{y=0} \qquad \underline{y^2=2x^2} \longrightarrow y = \pm\sqrt{\frac{2}{3}}$.

$\underline{x=\pm 1}$

$(1,0)$ $\quad \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \quad \left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

$(-1,0)$ $\quad \left(\frac{-1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) \quad \left(\frac{-1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

$f(-1,0) = 0 \quad f\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) = \frac{2}{3\sqrt{3}} = f\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

$f(1,0) = 0 \quad f\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) = \frac{-2}{3\sqrt{3}} = f\left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$

Max value : $\frac{2}{3\sqrt{3}}$ Min value: $\frac{-2}{3\sqrt{3}}$.

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e.g. Three variables.

$f(x,y,z) = xyz \qquad g(x,y,z) = 2x+2y+z-108.$

$\nabla f = \langle yz, xz, xy \rangle \quad \nabla g = \langle 2, 2, 1 \rangle$

$yz = 2\lambda$
$xz = 2\lambda$
$xy = \lambda$
$2x+2y+z = 108.$

$\longrightarrow yz = xz \longrightarrow xz - yz = 0$

$z(x-y) = 0$

$z = 0$ or $x = y$

$z=0$:
$\downarrow$
$2\lambda = 0$
$\lambda = 0$
$xy = 0$
$\left(\begin{array}{l} x = 0 \text{ or} \\ y = 0. \end{array}\right)$
Forget it.
NOT A SOLUTION

$x=y$:
$x^2 = \lambda$
$xz = 2\lambda$
$xz = 2x^2$
$z = 2x$

$\downarrow$

$2x+2x+2x = 108$

$6x = 108$

$\boxed{\begin{array}{l} x = 18 \\ y = 18 \\ z = 36 \end{array}}$