Tangent Planes

Surfaces:

① $f(x,y,z) = c$ (usu $f(x,y,z)=0$)
Level surface of $w = f(x,y,z)$.
Tangent plane at P_0 on surface has normal vector $\nabla f(P_0)$.

② $z = f(x,y)$. Graph of $f(x,y)$.
Note: If $F(x,y,z) = f(x,y) - z$, then surface is given by $F(x,y,z) = 0$.
Tangent plane at $P_0 = (x_0, y_0, f(x_0,y_0))$ has normal vector

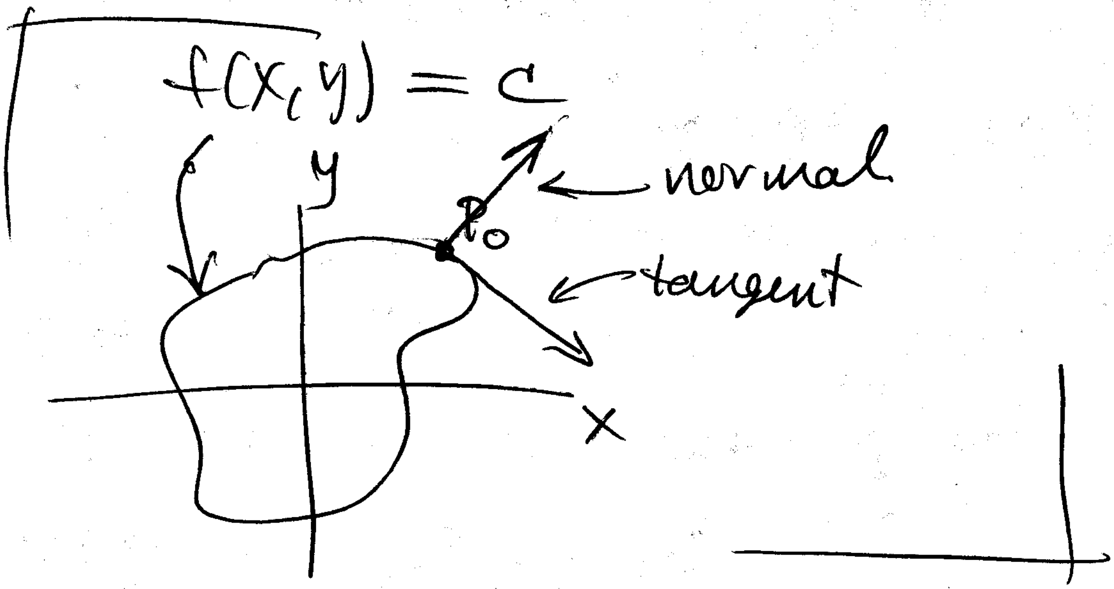
$$\nabla F(P_0) = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = f_x \vec{i} + f_y \vec{j} + (-1)\vec{k}$$

#(8) $z = \underbrace{\ln(1+xy)}_{f(x,y)}$ $\quad P_0 = (1, 2, \ln 3)$

$$\vec{n} = f_x(1,2)\vec{i} + f_y(1,2)\vec{j} - \vec{k} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \vec{k}$$

$$f_x = \frac{y}{1+xy} \qquad f_y = \frac{x}{1+xy}$$

$$f_x(1,2) = \frac{2}{3} \qquad f_y(1,2) = \frac{1}{3}.$$



T.P. : $\frac{2}{3}(x-1)+\frac{1}{3}(y-2)-(z-\ln 3)=0$

$$\frac{2}{3}x-\frac{2}{3}+\frac{1}{3}y-\frac{2}{3}-z+\ln 3=0$$

$$\frac{2}{3}x+\frac{1}{3}y-z=-\left(\ln 3-\frac{4}{3}\right)$$ ← one way to rewrite.

Another way:

$$z=\underbrace{(\ln 3)+\frac{2}{3}(x-1)+\frac{1}{3}(y-2)}_{\text{linear function of } x, y}$$

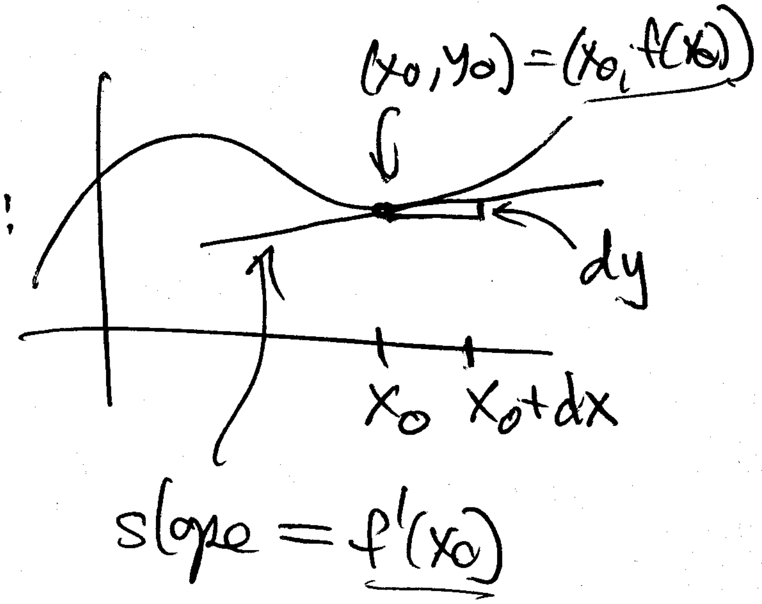
graph is a plane.

Recall: $y = f(x)$

(a) linearization of f at x_0:

$y - f(x_0) = f'(x_0)(x - x_0)$

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0).}$$



Point: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ x near x_0

(b) Differentials (and approximations)

$dy = f'(x_0)\,dx.$

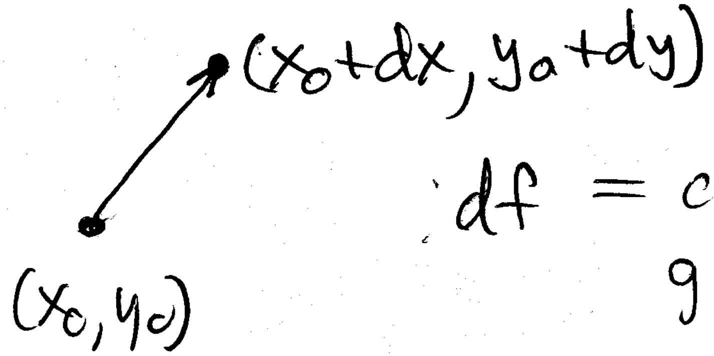
$dy \approx$ change in f when x goes from x_0 to $x_0 + dx$.

In several variables:

(a) linearization: $f(x, y)$ $P_0 = (x_0, y_0, f(x_0, y_0))$

$$z = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x,y)\,:\ \text{linearization of } f \text{ at } P_0.}$$

(b) Differentials:



df = change in $L(x,y)$ given change in (x_0,y_0).

$$\begin{aligned} df &= L(x_0+dx, y_0+dy) - L(x_0,y_0) \\ &= \cancel{f(x_0,y_0)} + f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy - \cancel{f(x_0,y_0)} \\ &= f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy. \end{aligned}$$

Short hand notation:

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.}$$

Point: $df \approx$ actual change in f as (x_0,y_0) goes to (x_0+dx, y_0+dy).

eg #26) $f(x,y) = \dfrac{x+y}{x-y}$ $\quad (3,2)$

estimate $f(2.95, 2.05)$

$(3-.05, 2+.05)$

$(3,2)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad \begin{aligned} dx &= -.05 \\ dy &= .05 \end{aligned}$$

$$\frac{\partial f}{\partial x} = \frac{(x-y)(1)-(x+y)(1)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

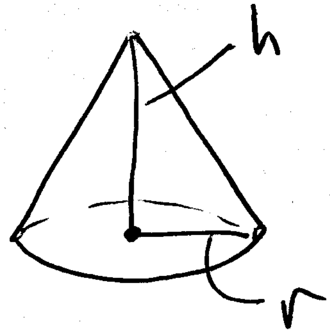
$$\frac{\partial f}{\partial y} = \frac{(x-y)(1)-(x+y)(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial f}{\partial x}(3,2) = -4 \qquad \frac{\partial f}{\partial y}(3,2) = 6$$

$$df = (-4)(-.05) + (6)(.05) = .5$$

$$f(2.95, 2.05) \approx f(3,2) + .5 = 5.5.$$

#32)



$$V = \frac{\pi r^2 h}{3}$$

$r: 6.5 \rightarrow 6.6$

$h: 4.20 \rightarrow 4.15$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh$$

$r = 6.5$

$h = 4.2$

$dr = .1$

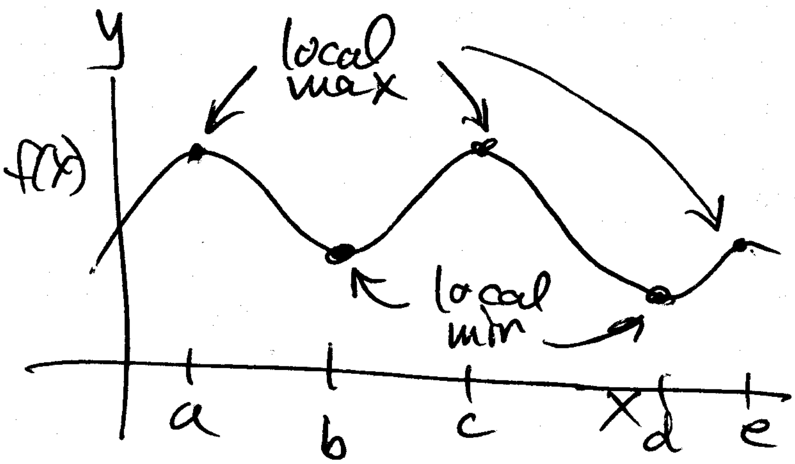
$dh = -.05$

$$dV = \frac{2\pi(6.5)(4.2)}{3}(.1) + \frac{\pi(6.5)^2}{3}(-.05)$$

$$\approx 5.72 + (-2.21) = 3.51 \text{ (units)}^3$$

12.8 Extreme Values Problems.

<u>Recall</u>: $y = f(x)$

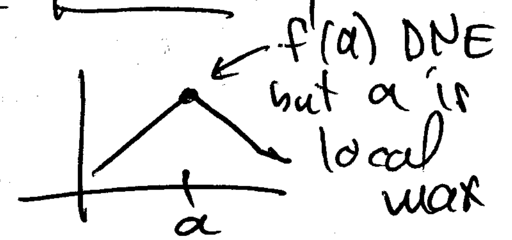


How do you find local max/min:

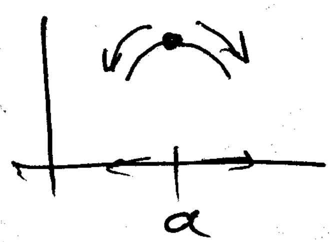
Solve $f'(x) = 0$.

e.g. $f'(a) = 0$, a is a <u>critical point</u>.

or if $f'(a)$ does not exist:

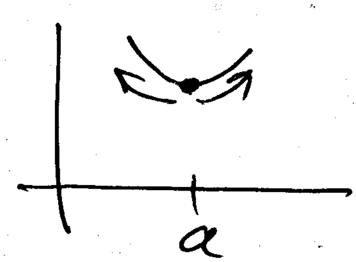


Three possibilities:



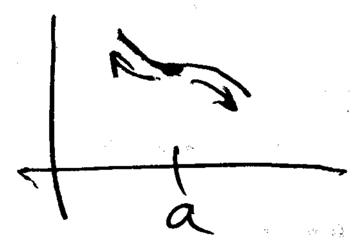
local max

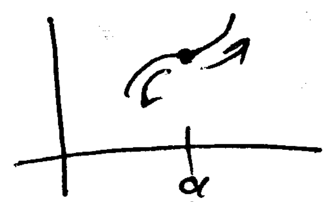
$f''(a) < 0$



local min

$f''(a) > 0$

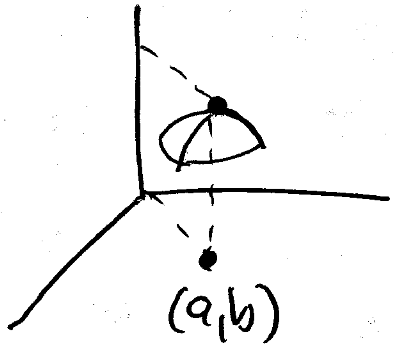




neither

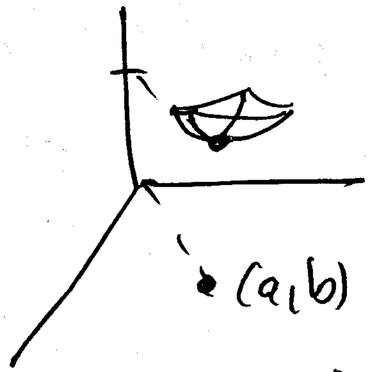
$f''(a) = 0$.

$z = f(x,y)$



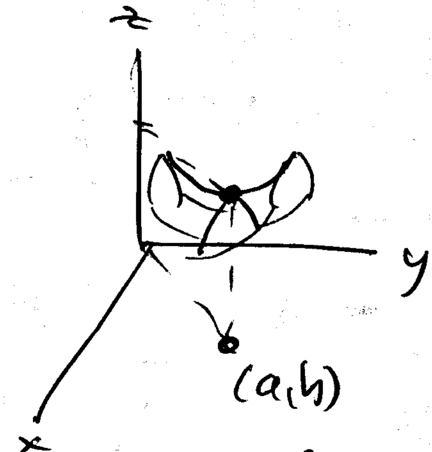
local max

$\nabla f(a,b) = \vec{0}$



local min

$\nabla f(a,b) = \vec{0}$



saddle point.

$\nabla f(a,b) = \vec{0}$

<u>Fact</u>: If f has a local min or max at (a,b) then all directional derivatives are zero at that point so $\nabla f(a,b) = 0$. or $\nabla f(a,b)$ is not defined. (a,b) is a <u>critical point</u> of f.

eg: $f(x,y) = x^2+3xy+3y^2-6x+3y-6$

$$\nabla f(x,y) = f_x \vec{i} + f_y \vec{j}$$

$$= (2x+3y-6)\vec{i} + (3x+6y+3)\vec{j}$$

$\nabla f = \vec{0}$ means $f_x = f_y = 0$.

$$2x+3y-6=0 \longrightarrow (2x+3y=6)(-2)$$

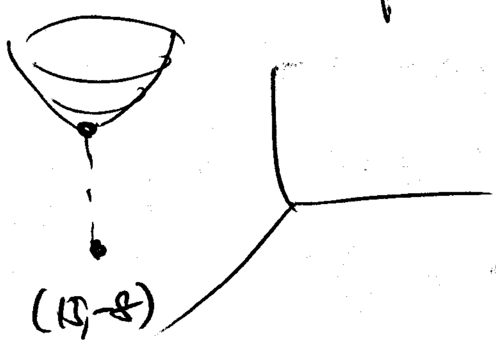
$$3x+6y+3=0 \longrightarrow 3x+6y=-3$$

$$\longrightarrow -x = -15 \longrightarrow \boxed{x=15}$$

$$\longrightarrow 30+3y=6 \longrightarrow \boxed{y=-8}$$

$$3y=-24$$

Critical point: $(15,-8)$.



e.g. $f(x,y) = 8x^3 + y^3 + 6yx$

Find critical pts:

$f_x = 24x^2 + 6y$

$f_y = 3y^2 + 6x$

$24x^2 + 6y = 0$

$3y^2 + 6x = 0$

$4x^2 + y = 0$

$y^2 + 2x = 0$

$y = -4x^2$

$(-4x^2)^2 + 2x = 0$

$16x^4 + 2x = 0$

$2x(8x^3 + 1) = 0$

$x = 0 \quad x = -\frac{1}{2}$

$y = 0 \quad y = -1$

Critical pts:

$(0,0) \quad (-\frac{1}{2}, -1)$.