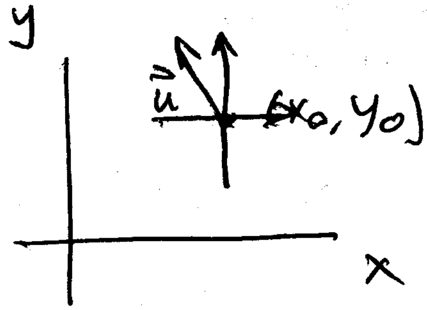
12.6 Directional Derivatives + Gradient

A. Directional derivatives

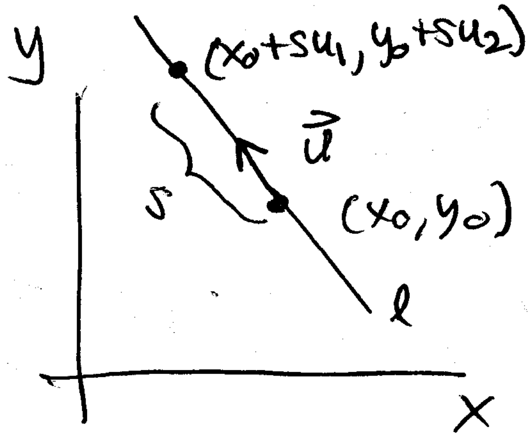
$z = f(x,y)$



Know: Rate of change in x-dir : $f_x(x_0, y_0)$

" " " in y-dir : $f_y(x_0, y_0)$

Idea:



$\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ is a unit vector.

Parametrize line:

$x = x_0 + su_1$

$y = y_0 + su_2$

Restricted to line l, $z = f(x,y)$ is a function of one variable, called s. Rate of change will be:

$$D_{\vec{u}} f(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0, y_0)}{s}$$

How do we compute this?

If $z = f(x,y)$ $\quad x = x_0 + s u_1 \quad y = y_0 + s u_2$

then z is a function of s, and

$$D_{\vec{u}} f(x_0, y_0) = \left.\frac{dz}{ds}\right|_{s=0}.$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

$$= \frac{\partial z}{\partial x} u_1 + \frac{\partial z}{\partial y} u_2$$

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\, u_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, u_2 .}$$

e.g. $f(x,y) = x^2 + xy$, $P_0 = (1,2)$

$\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

Find $D_{\vec{u}} f(P_0)$

$$D_{\vec{u}} f(1,2) = f_x(1,2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1,2)\frac{1}{\sqrt{2}}$$

$$= (4)\left(\frac{1}{\sqrt{2}}\right) + (1)\frac{1}{\sqrt{2}}$$

$$= \frac{5}{\sqrt{2}}$$

$f_x = 2x + y$

$f_y = x$

B. Gradient.

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2$$

$$= \underbrace{\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle}_{\text{gradient of } f \text{ at } (x_0, y_0)} \cdot \langle u_1, u_2 \rangle$$

$$= \nabla f(x_0, y_0) \cdot \vec{u}.$$

eg #13) $F(x, y) = e^{-x^2 - 2y^2}$ $\quad P = (-1, 2)$

Find $\nabla F(-1, 2)$. $\quad F_x = -2x\, e^{-x^2 - 2y^2}$

$$F_y = -4y\, e^{-x^2 - 2y^2}$$

$$\nabla F(x, y) = \langle -2x\, e^{-x^2 - 2y^2}, -4y\, e^{-x^2 - 2y^2} \rangle$$

$$= -2\, e^{-x^2 - 2y^2} \langle x, 2y \rangle$$

$$\nabla F(-1, 2) = -2\, e^{-9} \langle -1, 4 \rangle.$$

Find $D_{\vec{u}} F(-1, 2)$ ~~when~~ in direction $\vec{v} = \vec{i} + 5\vec{j}$. Need unit vector parallel to $\vec{v}$ (same direction) $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 5\vec{j}}{\sqrt{26}} = \frac{1}{\sqrt{26}}\vec{i} + \frac{5}{\sqrt{26}}\vec{j}$

$$D_{\vec{u}} F(-1, 2) = -2e^{-9} \langle \underset{3}{-1}, 4 \rangle \cdot \left\langle \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right\rangle = -2e^{-9}\left(\frac{19}{\sqrt{26}}\right).$$

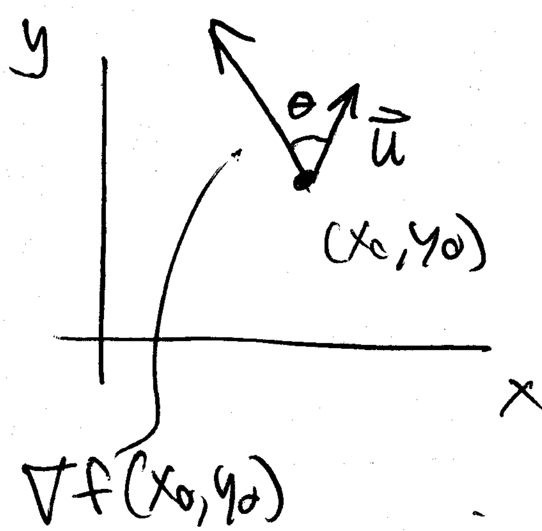
eg $f(x,y,z) = \frac{xy}{z}$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$= \left\langle \frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2} \right\rangle$$

C. Properties of gradient.

1. ∇f is a vector-valued function of 2 or 3 variables. Called a vector field.

2. $\nabla f(x_0,y_0)$ has <u>magnitude</u> and <u>direction</u>



$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$$
$$= |\nabla f(x_0,y_0)|\,\overset{1}{\cancel{|\vec{u}|}} \cos\theta$$
$$= |\nabla f(x_0,y_0)| \cos\theta$$

If $\theta = 0$ then $\cos\theta = 1$ is as large as it gets, if $\theta = \pi$ then $\cos\theta = -1$ is as <u>small</u> as it gets

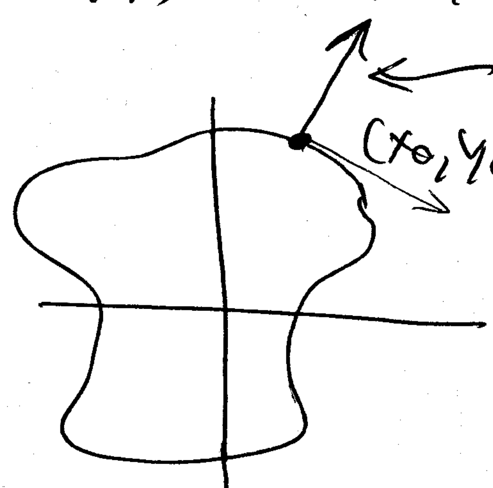
<u>Fact</u>: $D_{\vec{u}} f(x_0,y_0)$ is maximized when $\vec{u} = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$ and has value $|\nabla f(x_0,y_0)|$

and is minimized in direction $-\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$

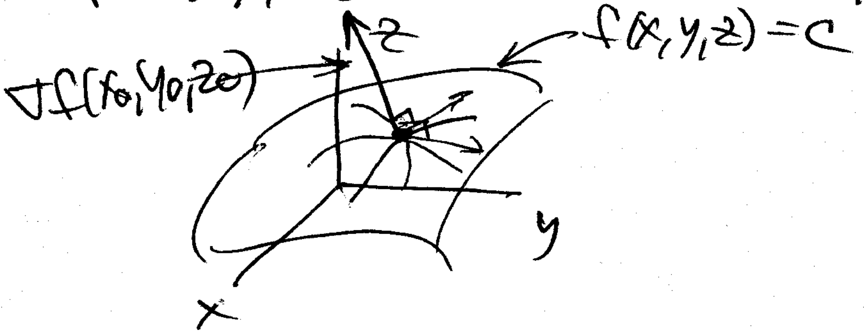
and has value $-|\nabla f(x_0,y_0)|$.

3. If $\theta = \frac{\pi}{2}$ then $D_{\vec{u}} f(x_0,y_0) = 0$, i.e. $\nabla f(x_0,y_0)$ is perpendicular to the level curve of f at (x_0,y_0)

$z = f(x,y)$ $\qquad$ $f(x,y) = c = f(x_0,y_0)$



In 3-variables: $\nabla f(x_0,y_0,z_0)$ is normal to the level surface $f(x,y,z) = c = f(x_0,y_0,z_0)$



<u>eg</u>

$g(x,y,z) = xe^y + z^2 \quad P_0 = (1, \ln 2, 1/2)$

$\nabla g(x,y,z) = e^y \vec{i} + xe^y \vec{j} + 2z \vec{k}.$

$\nabla g(1, \ln 2, 1/2) = 2\vec{i} + 2\vec{j} + \vec{k}$

max rate of change of g at P_0

$|\nabla g(1, \ln 2, 1/2)| = (4+4+1)^{1/2} = 3.$

direction of max rate of change of g at P_0

$$\frac{\nabla g(1, \ln 2, 1/2)}{|\nabla g(1, \ln 2, 1/2)|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$$

$g(1, \ln 2, 1/2) = \frac{9}{4}$

So $2\vec{i} + 2\vec{j} + \vec{k}$ is perpendicular to the level surface $xe^y + z^2 = 9/4$ at the point $P_0 = (1, \ln 2, 1/2)$.

12.7 Tangent Planes & Differentials

A. Tangent Planes.

Given a surface: $f(x,y,z)=c$ (i.e. level surface of f) or as graph of $z=f(x,y)$.

Given P on the surface $f(x,y,z)=c$ we know $\nabla f(P)$ is perpendicular to the surface.

The tangent plane to surface $f(x,y,z)=c$ at point P is the plane through P with normal vector $\nabla f(P)$.

eg $x^2+y^2-z^2=18$ $\quad P=(3,5,-4)$

$f(x,y,z)=x^2+y^2-z^2$ (surface is given by $f(x,y,z)=18$)

Find tangent plane.

$\vec{n}=\nabla f(3,5,-4)$ $\quad \nabla f=\langle 2x,2y,-2z\rangle$

$=\langle 6,10,8\rangle$ $\quad$ Plane: $\cancel{6}^{3}(x-3)+\cancel{10}^{5}(y-5)+\cancel{8}^{4}(z+4)=0$

$3x-9+5y-25+4z+16=0$

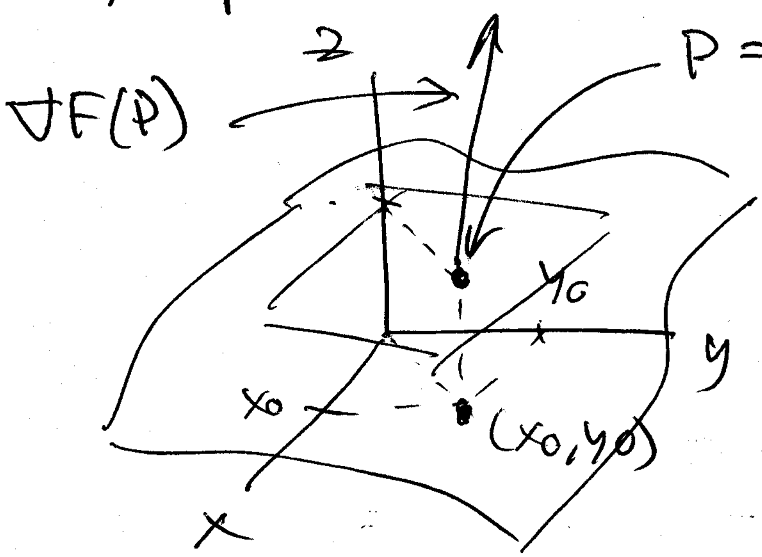
$\boxed{3x+5y+4z=18}$

B. Differentials

Another way to denote ~~a~~ some surfaces is as the graph of $z = f(x,y)$.

We can also write $F(x,y,z) = f(x,y) - z = 0$

<u>Problem</u>: Find the tangent plane to graph of $f(x,y)$ at (x_0, y_0).



$$\nabla F(P) = \langle f_x(P), f_y(P), -1 \rangle$$

Equation of TP:

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + (-1)(z - f(x_0,y_0)) = 0.$$

Solving for z

$$z = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{\text{linear function in } (x,y)}.$$

This function is the <u>linearization</u> of $f(x,y)$ at (x_0,y_0) and is denoted $L(x,y)$.