Partial Derivatives

$z = f(x,y)$

① rate of change of f depends on direction
② pick two preferred directions (x-direction and y-direction)
③ for each direction define rate of change using difference quotients as usual

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e.g. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

$\frac{\partial f}{\partial x} = 5y - 14x + 3$ $\qquad$ $\frac{\partial f}{\partial y} = 5x - 2y - 6$

↑ treat y as const. $\qquad$ ↑ treat x as const.

e.g. $f(x,y) = (2x - 3y^2)^3$

$$\frac{\partial f}{\partial x} = 3(2x - 3y^2)^2(2) = 6(2x - 3y^2)^2$$

$$\frac{\partial f}{\partial y} = 3(2x - 3y^2)^2(-6y) = -18y(2x - 3y^2)^2$$

Higher order derivatives.

Notation: $z = f(x,y)$

$$\frac{\partial f}{\partial x} \text{ or } f_x \qquad \frac{\partial f}{\partial y} \text{ or } f_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$(f_x)_x = f_{xx} \qquad (f_y)_y = f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_x)_y = f_{xy} \qquad (f_y)_x = f_{yx}$$

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eg: $f(x,y) = (2x-3y^2)^3$

Find $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(6(2x-3y^2)^2\right)$

$$= 12(2x-3y^2)(2) = 24(2x-3y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-18y(2x-3y^2)^2\right)$$

$$= (-18y)\left(2(2x-3y^2)(-6y)\right) + (-18)(2x-3y^2)^2$$

$$= (2x-3y^2)\left(216y^2 - 18(2x-3y^2)\right)$$

$$= (2x-3y^2)(270y^2 - 36x).$$

Find $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(6(2x-3y^2)^2\right)$

$= 12(2x-3y^2)(-6y) = -72y(2x-3y^2)$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-18y(2x-3y^2)^2\right)$

$= (-18y)(2(2x-3y^2)(2)) = -72y(2x-3y^2)$

<u>e.g.</u> $f(x,y) = e^{xy}\ln(y)$

$f_x = ye^{xy}\ln(y)$ $\quad f_y = e^{xy}\cdot\frac{1}{y} + xe^{xy}\ln(y)$

$f_{xx} = y^2e^{xy}\ln(y)$ $\quad f_{yy} = e^{xy}\cdot\frac{-1}{y^2} + xe^{xy}\frac{1}{y}$

$+ xe^{xy}\cdot\frac{1}{y} + x^2e^{xy}\ln(y)$

$= -\frac{e^{xy}}{y^2} + 2xe^{xy}\cdot\frac{1}{y} + x^2e^{xy}\ln(y)$

$f_{xy} = \cancel{y}e^{xy}\cdot\frac{1}{\cancel{y}} +$

$(xye^{xy} + e^{xy})\ln(y)$

$= e^{xy} + xye^{xy}\ln(y)$

$+ e^{xy}\ln(y).$

$f_{yx} = \cancel{y}e^{xy}\cdot\frac{1}{\cancel{y}} + (xye^{xy} + e^{xy})\ln(y)$

$= e^{xy} + xye^{xy}\ln(y) + e^{xy}\ln(y).$

$$f_{xyx} = (f_{xy})_x$$
$$= ye^{xy} + (xy^2e^{xy} + ye^{xy})\ln(y) + ye^{xy}\ln(y).$$
$$= ye^{xy} + xy^2e^{xy}\ln(y) + 2ye^{xy}\ln(y).$$

$$f_{xxy} = (f_{xx})_y$$
$$= y^{\cancel{2}}e^{xy}\cdot\frac{1}{\cancel{y}} + (xy^2e^{xy} + 2ye^{xy})\ln(y).$$
$$= ye^{xy} + xy^2e^{xy}\ln(y) + 2ye^{xy}\ln(y).$$

$$f_{yyx} = (f_{yy})_x = \frac{-\cancel{y}e^{xy}}{y^{\cancel{2}}} + (2xye^{xy} + 2e^{xy})\frac{1}{y}$$
$$+ (x^2ye^{xy} + 2xe^{xy})\ln(y).$$

12.5 Chain rule.

Recall: 1-variable

$w = f(x) \quad x = g(t) \longrightarrow w = f(g(t)) = (f \circ g)(t)$

$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} \qquad \frac{dw}{dt} = f'(g(t))g'(t)$

Idea: $dw = \left(\frac{dw}{dt}\right) dt \qquad dx = \left(\frac{dx}{dt}\right) dt$

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$dw = \left(\frac{dw}{dx}\right)\left(\frac{dx}{dt}\right) dt \longleftarrow dw = \left(\frac{dw}{dx}\right) dx$

Now let $z = f(x, y) \quad x = g(t) \quad y = h(t)$

z is ultimately a function of t

$z = f(x(t), y(t))$.

Want $\frac{dz}{dt}$ or $dz = \underbrace{\left(\frac{dz}{dt}\right)}_{?} dt \qquad dx = \left(\frac{dx}{dt}\right) dt$

$dy = \left(\frac{dy}{dt}\right) dt$

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$= \left(\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right) dt$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

$w = f(x, y, z) \qquad x = g(t) \quad y = h(t) \quad z = p(t)$

Find $\frac{dw}{dt}$.

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

$z = f(x, y) \qquad x = g(r, s) \quad y = h(r, s)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

e.g. $w = x^2 + y^2 \qquad x = \cos t + \sin t$
$\qquad\qquad\qquad\qquad y = \cos t - \sin t$

Find $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$$= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$$

$$= 2(\cos t + \sin t)(-\sin t + \cos t) + 2(\cos t - \sin t)(-\sin t - \cos t)$$

$$= 2(-\cancel{\cos t \sin t} - \cancel{\sin^2 t} + \cancel{\cos^2 t} + \cancel{\cos t \sin t})$$
$$+ 2(-\cancel{\cos t \sin t} - \cancel{\cos^2 t} + \cancel{\sin^2 t} + \cancel{\sin t \cos t}) = 0$$

Note that $w = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2$

so w is const. as a function of t.

<u>eg</u> #22 p827)

$w = (x^2+y^2+z^2)^{1/2} \quad x = st \quad y = rs \quad z = rt$

Find w_r, w_s, w_t

$$w_r = \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(\tfrac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}x\right)(0) + y(x^2+y^2+z^2)^{-1/2}(s)$$

$$+ z(x^2+y^2+z^2)^{-1/2}(t)$$

$$= \frac{ys+zt}{(x^2+y^2+z^2)^{1/2}} = \frac{rs^2+rt^2}{(s^2t^2+r^2s^2+r^2t^2)^{1/2}} \quad //$$