Level curves and level surfaces.

$z = f(x,y)$. The level curves of $f(x,y)$ are the curves $f(x,y) = c$ for different values of c.

$$z = (x-4y^2)^{1/2}$$

Level curves are parabolas opening to the right.

$$(x-4y^2)^{1/2} = c$$
$$x = 4y^2 + c^2$$

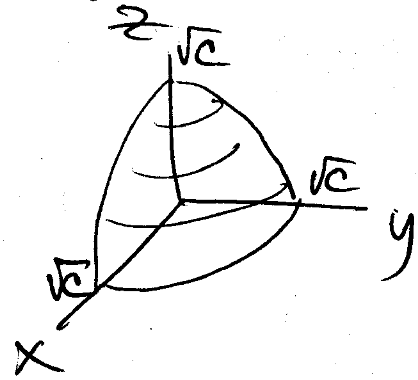
$w = f(x,y,z)$. Can't visualize the graph of f.

The level surfaces of $f(x,y,z)$ are the surfaces $f(x,y,z) = c$ for different values of c.

e.g. $f(x,y,z) = x^2+y^2+z^2$

Level surfaces are spheres of radius $\sqrt{c}$

$$x^2+y^2+z^2 = c$$



Think of: $z = f(x,y)$

z as height above sea level on a hillside.

$w = f(x,y,z)$ w as temperature at the point (x,y,z)

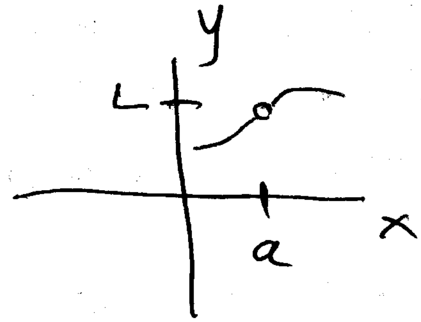
12.3 Limits and Continuity

Recall: $y = f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $\Delta x = 0$ then you get $\frac{0}{0}$. Limits were invented to give meaning to expressions that evaluate to $0/0$.

Recall: $\lim_{x \to a} f(x) = L$



we need $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$. One-sided limits must agree.

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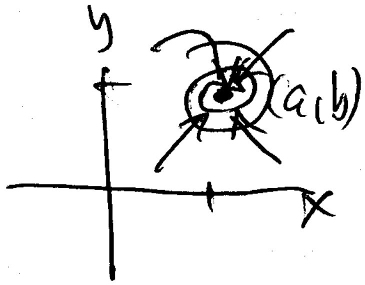
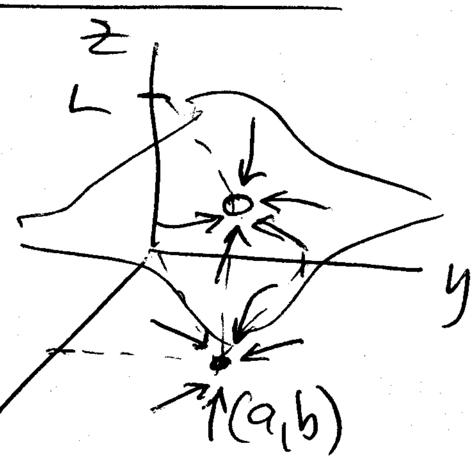
look at:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$





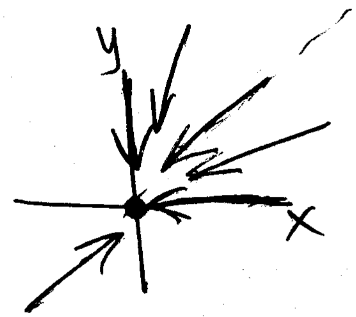
For limit to exist, need to consider

(1) infinitely many directions

(2) even more different paths.

Point: Evaluating $\lim_{(x,y)\to(a,b)} f(x,y) = L$ is much harder than in 1-variable. In particular if an expression evaluates to $\frac{0}{0}$, it is much harder to deal with.

e.g. $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ Evaluate at $(0,0)$: get $\frac{0}{0}$.



Check some paths:

① Path: $y=0$. Limit becomes
$$\lim_{x\to 0} \frac{2x\cdot 0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

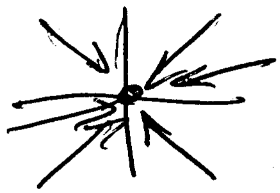
② Path: $x=0$. Limit becomes
$$\lim_{y\to 0} \frac{2\cdot 0\cdot y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

③ Path $y=x$. Limit becomes
$$\lim_{x\to 0} \frac{2x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1.$$

④ Path: $y=mx$
$$\lim_{x\to 0} \frac{2x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{2m\cancel{x^2}}{(1+m^2)\cancel{x^2}} = \frac{2m}{1+m^2}$$

eg $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ (evaluates to $\frac{0}{0}$)

① Try $y = mx$ $(m \neq 0)$

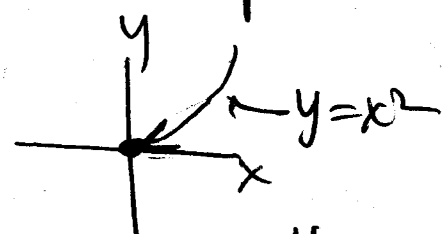


$$\lim_{x\to 0} \frac{2x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{2mx^{\cancel{3}}}{x^{\cancel{4}2}+m^2\cancel{x^2}} = \lim_{x\to 0} \frac{2mx}{x^2+m^2}$$

$= 0$

For all straight line paths, limit is 0.

② Try $y = x^2$



$$\lim_{x\to 0} \frac{2x^2 \cdot x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1.$$

Limit does not exist.

eg $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ Try any path

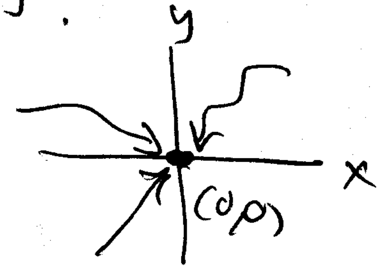
$y = mx$, $y = x^2$, $y = x^3$

$y = x^{1/2}$, etc...

You will always get 0.

Maybe the limit really does exist. How would you show this?

Trick: Try polar coordinates.



ANY PATH HAS $r \to 0$.

$$x = r\cos\theta \qquad x^2+y^2=r^2$$
$$y = r\sin\theta$$

Limit becomes:

$$\lim_{r\to 0} \frac{4r\cos\theta\, \cancel{r^2}\sin^2\theta}{\cancel{r^2}}$$

$$= \lim_{r\to 0} \underbrace{4r}_{\text{going to } 0} \underbrace{\cos\theta\sin^2\theta}_{\text{limited to values in } [-1,1]} = 0.$$

Point: $z = f(x,y)$

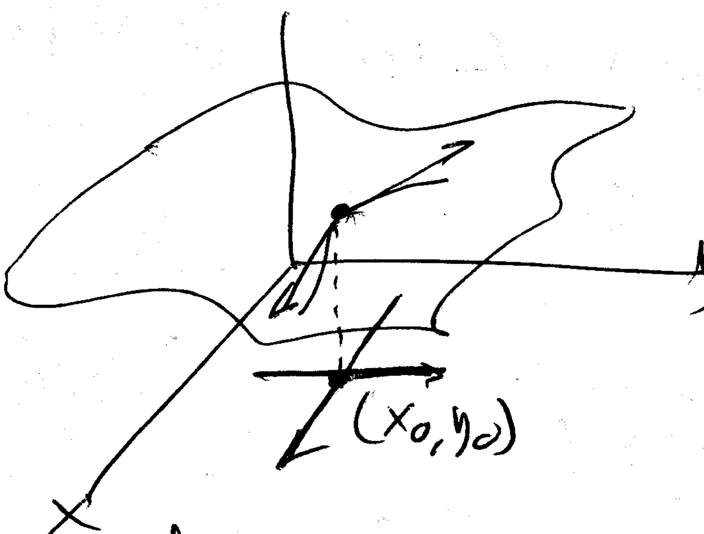
$$f'(x,y) = \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{(\Delta x^2 + \Delta y^2)^{1/2}}$$

does not work. It rarely exists. Too restrictive a definition.

12.4 Partial Derivatives.

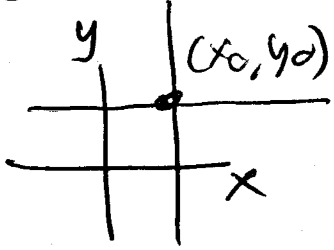
Want: 1) notion of rate of change
and 2) notion of "slope" of "tangent line"

for $z = f(x, y)$ or $w = f(x, y, z)$.



Different directions give different rates of change.

We choose 2 special directions and find slopes in these directions.



Rate of change in x-direction at (x_0, y_0)

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0 \cancel{+\Delta x}, y_0)}{\Delta x}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$f(x,y) = x^2 - xy + y^2$

Find $\frac{\partial f}{\partial x}$. $\frac{\partial f}{\partial x} = 2x - y$ ← Think of y as const.

$$\frac{\partial f}{\partial y} = 0 - x + 2y = 2y - x.$$

Think of x as const.