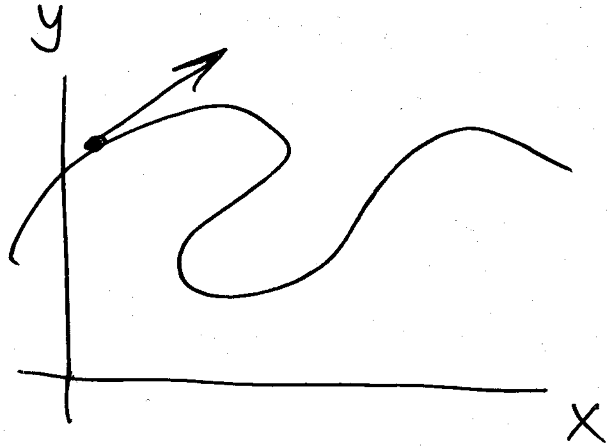
Idea behind curvature



$\vec{r}(t)$

$\vec{r}\,'(t)$ – velocity

$|\vec{r}\,'(t)|$ – speed.

$\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \vec{T}(t)$ – unit tangent vector.

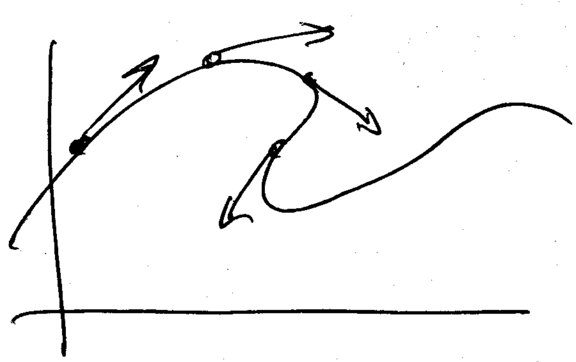
$\vec{r}(t) = \vec{R}(s)$ $\qquad s = \int_a^t |\vec{r}\,'(u)|\,du$

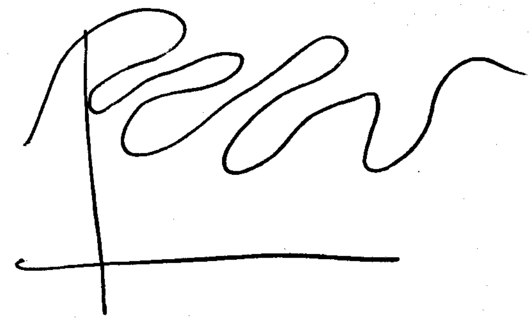
↑ arclength parameter

$\left|\dfrac{d\vec{R}}{ds}\right| = 1$

$\dfrac{d\vec{R}}{ds}$ = unit tangent vector.

If $\vec{r}(t)$ has speed 1, so $|\vec{r}\,'(t)| = 1$





Measuring curvature: Rate of change of direction of $\vec{r}\,'(t)$ when $|\vec{r}\,'(t)| = 1$

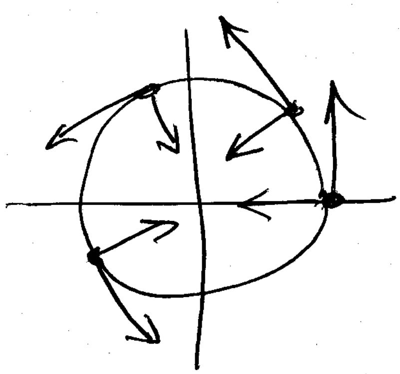
So definition is $K(s) = \cancel{\left|\frac{d}{dt}\vec{T}(t)\right|}\ \left|\dfrac{d}{ds}\vec{T}(s)\right|$

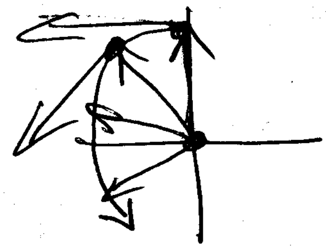
With respect to original parameter t:

$$\kappa(t) = \frac{1}{|\vec{r}\,'(t)|}\left|\frac{d}{dt}\vec{T}(t)\right|$$

measures how "bendy" the curve is at parameter t.

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} \quad |\vec{r}\,'(t)| = 1$

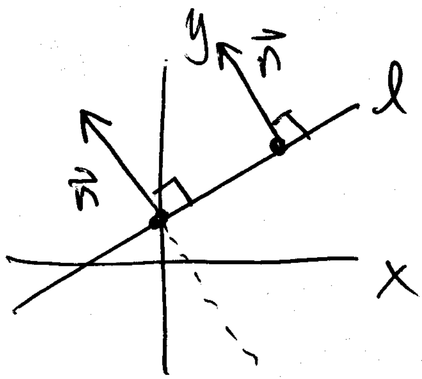




$$\frac{d}{dt}\vec{T}(t) = \frac{d}{dt}\vec{r}\,'(t) = \vec{r}\,''(t).$$

$\kappa(t) = 1$ (in general, $\frac{1}{\text{radius}}$)

12.1 Planes and Surfaces.



What determines a line?

① 2 points

② 1 point + $\underbrace{\text{slope}}$

measures "steepness" or "tilt" of line

Any line can be written

$ax+by=c. \longrightarrow y=\left(\frac{-a}{b}\right)x+\left(\frac{c}{b}\right)$

($b=0 \Leftrightarrow$ vertical line)

Find $\vec{n}$ perpendicular to l:

"slope" of $\vec{n}$ is $\frac{b}{a}$ so we can write $\vec{n}=\langle 1,\frac{b}{a}\rangle$

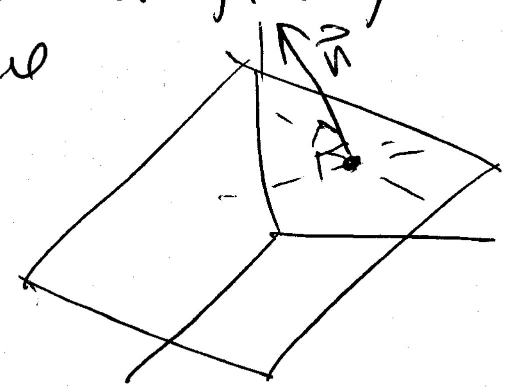
or, cleaning up a little, $\vec{n}=\langle a,b\rangle$.

What about planes?

What determines a plane?

① 3 points (non-colinear)

② 1 point and a vector giving "tilt" of plane



Any plane can be written

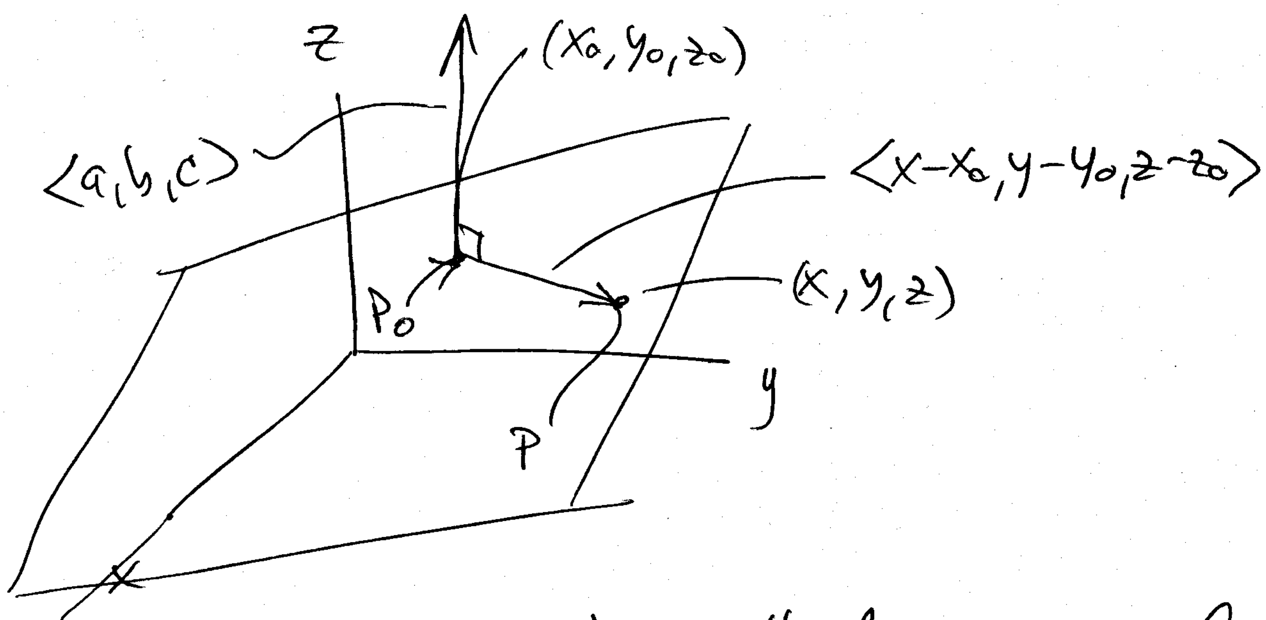
$$\boxed{ax+by+cz=d.}$$

If (x,y,z) and (x_0,y_0,z_0) are on plane P

then $ax+by+cz=d=ax_0+by_0+cz_0$

or $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

or $\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0.$



$\vec{n}=\langle a,b,c\rangle$ is called a normal vector to the plane.

eg $P_0=(2,-3,4)$ $\vec{n}=\langle -1,2,3\rangle$

Idea: If $P=(x,y,z)$ on the plane then

$\langle -1,2,3\rangle \cdot \langle x-2, y+3, z-4\rangle = 0$

$-1(x-2)+2(y+3)+3(z-4)=0$

$-x+2+2y+6+3z-12=0$

$-x+2y+3z-4=0$

$\boxed{-x+2y+3z=4}$

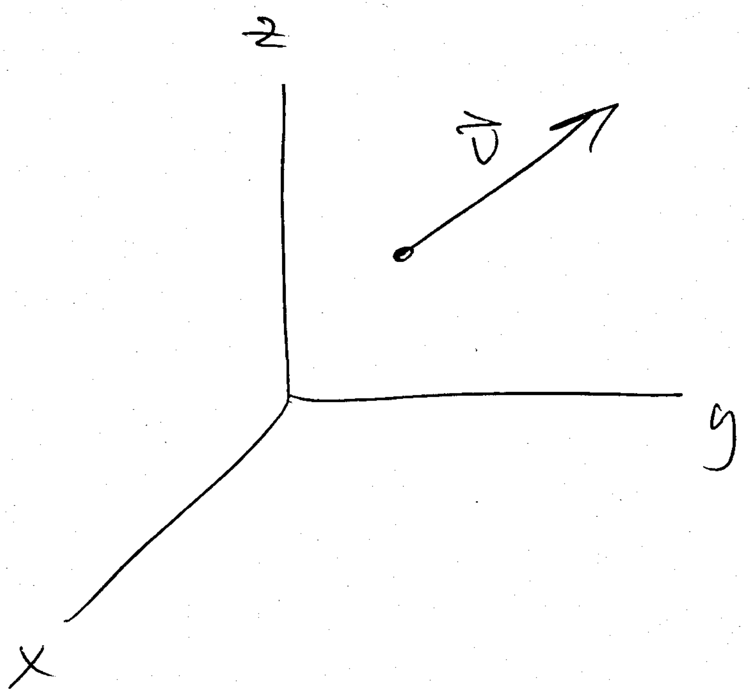
Different $\vec{n}$: Take: $\langle -2,4,6\rangle$

$-2(x-2)+4(y+3)+6(z-4)=0$

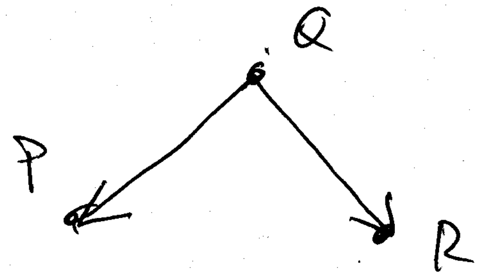
$-2x+4+4y+12+6z-24=0$

$\frac{1}{2}(-2x+4y+6z)=(8)\frac{1}{2}$

$-x+2y+3z=4$ (same as above)



eg 2 $P=(2,-1,3)$ $Q=(1,4,0)$ $R=(0,-1,5)$



$$\vec{n} = \overrightarrow{QR} \times \overrightarrow{QP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -5 & 5 \\ -1 & 5 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -5 & 5 \\ 5 & -3 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 5 \\ -1 & -3 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & -5 \\ -1 & 5 \end{vmatrix} \vec{k}$$

$$= -10\vec{i} - 8\vec{j} \cancel{+} -10\vec{k}$$

Take $\vec{n} = 5\vec{i} + 4\vec{j} + 5\vec{k}$ (above $\cdot\ -\frac{1}{2}$)

Take $Q=(1,4,0)$ as the point.

$$5(x-1)+4(y-4)+5(z-0)=0$$

$$5x-5+4y-16+5z=0$$

$$\boxed{5x+4y+5z=21}$$

Take P as our point, $P=(2,-1,3)$.

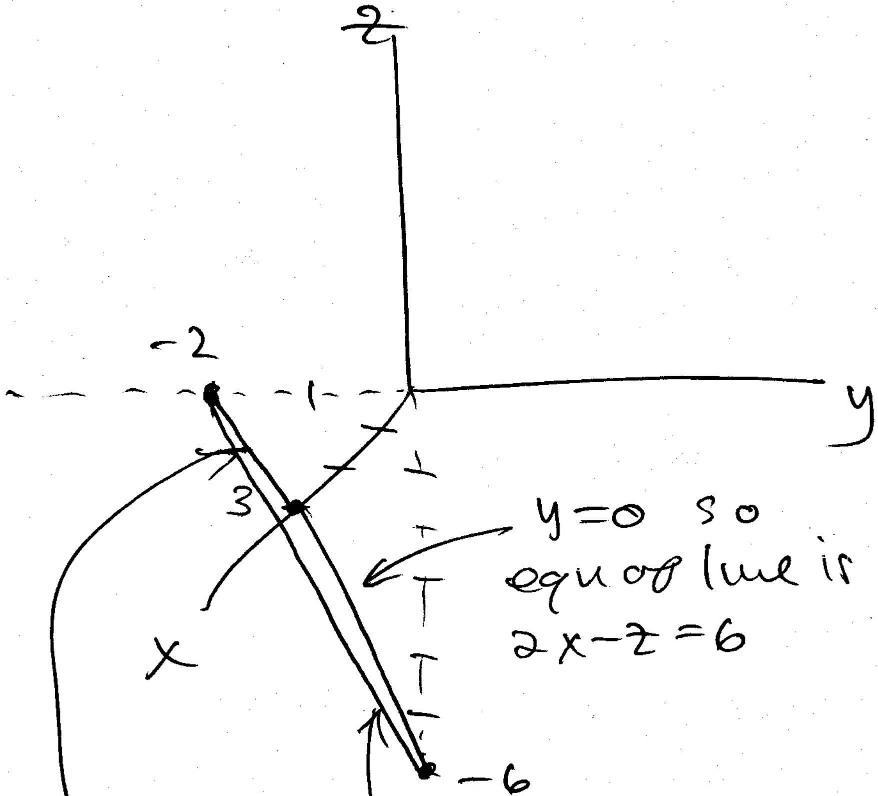
$$5(x-2)+4(y+1)+5(z-3)=0$$

$$5x-10+4y+4+5z-15=0$$

$$5x+4y+5z=21$$ (same as above)

e.g. 3 $2x-3y-z=6$

where does Q intersect coord axes.?



$y=0$ so equ of line is $2x-z=6$

$z=0$ here so equ of line is: $2x-3y=6$

$x=0$ so equ of line is $-3y-z=6$

x-axis: set $y=z=0$

$2x=6$

$x=3$

y-axis: $-3y=6$

$y=-2$.

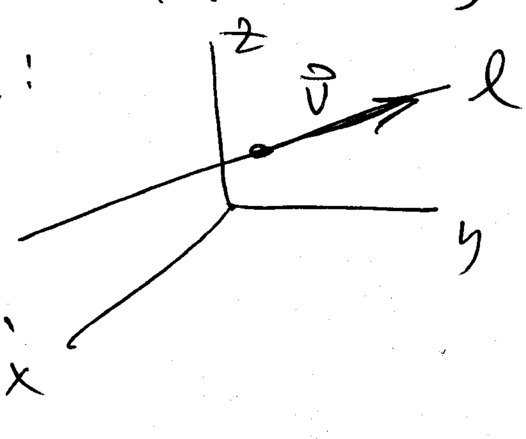
z-axis: $-z=6$

$z=-6$

e.g. 6) $Q: x+2y+z=5$ $R: 2x+y-z=7$

(if normal vectors are parallel, the planes are parallel and need not intersect)

Direction vector of line:
ℓ is in both planes
so $\vec{v}$ is perpendicular
to both normal vectors.



So we can take $\vec{v} = \langle 1,2,1\rangle \times \langle 2,1,-1\rangle$

$= -3\vec{i} + 3\vec{j} - 3\vec{k}$

To simplify could use $\boxed{\vec{v} = -\vec{i} + \vec{j} - \vec{k}}$

Point: $x + 2y + z = 5$

$2x + y - z = 7$

Let $x = 0$: $2y + z = 5$

$y - z = 7$

$3y = 12 \rightarrow \underline{y = 4}$

$4 - z = 7$ $\underline{z = -3}$

Can take $\boxed{P_0 = (0,4,-3)}$.

$\vec{r}(t) = \langle 0,4,-3\rangle + t\langle -1,1,-1\rangle$ //