Let $\vec{r}(t)$ be a smooth curve, $t \geq a$.

$$\boxed{\text{Arc Length function} \quad s(t) = \int_a^t |\vec{v}(u)|\, du}$$

$$\left(\vec{v}(u) = \frac{d\vec{r}}{dt}(u) \right)$$

<u>Ex 1</u> $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$ for $t \geq 0$, $a>0$, $b>0$
(helix)

$$\vec{v}(u) = \frac{d\vec{r}}{dt}(u) = \langle -a\sin t, a\cos t, b \rangle \Big|_{t=u}$$

$$\vec{v}(u) = \langle -a\sin u, a\cos u, b \rangle$$

$$s(t) = \int_0^t \left| \langle -a\sin u, a\cos u, b \rangle \right| du$$

$$s(t) = \int_0^t \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2}\, du$$

$$s(t) = \int_0^t \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2} \int_0^t du$$

$$\boxed{s(t) = \sqrt{a^2+b^2}\, t}$$

arc length function for helix.

Note: $\frac{d(s)}{dt} = \frac{d}{dt}\left(\int_a^t |\vec{v}(u)|du\right) = |\vec{v}(t)|$

$$\boxed{\frac{ds}{dt} = |\vec{v}(t)|} = \text{speed if } t \text{ is time.}$$

Definition: The parameter t in $\vec{r}(t)$ corresponds to arc length if

$$\frac{ds}{dt} = |\vec{v}(t)| = 1 \quad \text{for all } t \geq a$$

Ex 2 Given $\vec{r}(t) = \langle \cos(t^2), \sin(t^2), 0 \rangle$, $0 \leq t \leq \sqrt{\pi}$

Does the parameter t in $\vec{r}(t)$ correspond to arc length?

$$\vec{v}(t) = \vec{r}'(t) = \langle -2t \cdot \sin(t^2), 2t \cdot \cos(t^2), 0 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4t^2 \sin^2(t^2) + 4t^2 \cos^2(t^2) + 0^2}$$

$$\text{speed} = |\vec{r}'(t)| = \sqrt{4t^2} = 2t$$

Since $|\vec{v}(t)| = 2t \neq 1$ for all t in $[0, \sqrt{\pi}]$

t is not the parameter of arc length.

Ex2 cont.

Let's calculate the arc length function

for $\vec{r}(t) = \langle \cos(t^2), \sin(t^2), 0 \rangle$ $\quad 0 \le t \le \sqrt{\pi}$

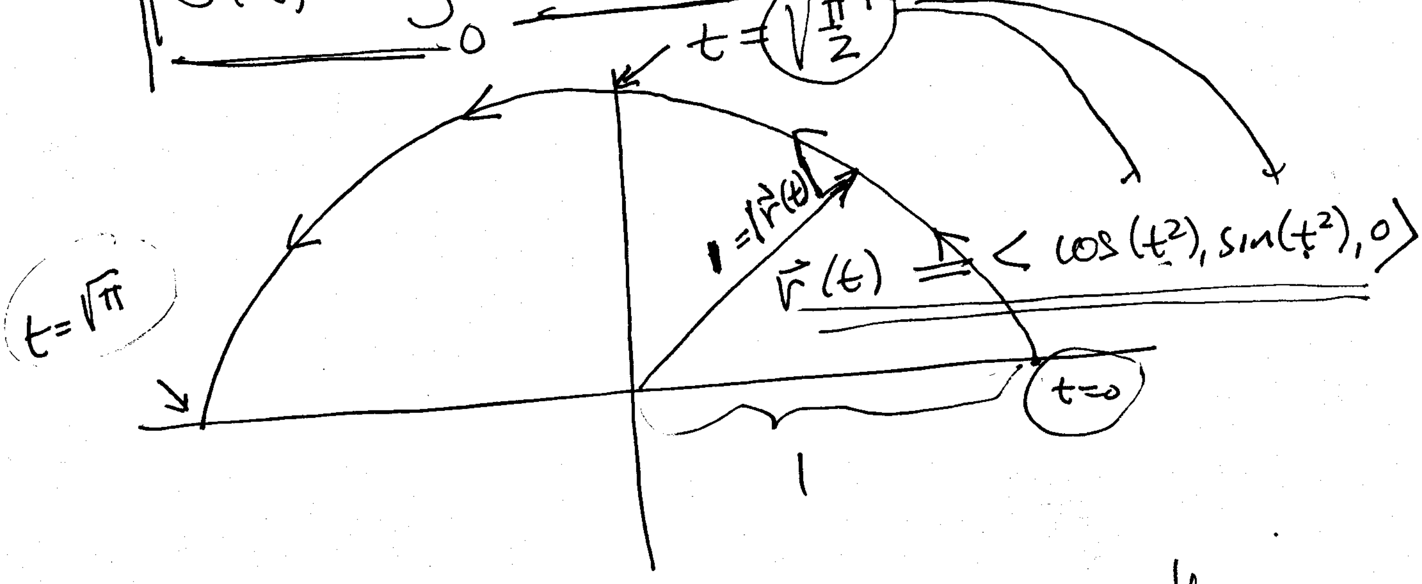
$$\boxed{s(t) = \int_0^t |\vec{v}(u)|\,du}$$

$|\vec{v}(t)| = 2t$

$$|\vec{v}(u)| = \left|\frac{d\vec{r}}{dt}(u)\right| = 2u$$

$$\boxed{s(t) = \int_0^t 2u\,du = t^2}$$

$0 \le t \le \sqrt{\pi}$



If we can calculate the integral in the arc length function, $s(t)$, and then solve for t, we can replace the parameter t in $\vec{r}(t)$ with s and obtain a curve with arc length as parameter.

Ex1 (return)

$$\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$$

$$\boxed{s = \sqrt{a^2+b^2}\, t} \qquad \left(|\vec{v}(t)| = \sqrt{a^2+b^2} \right)$$

$$t = \frac{s}{\sqrt{a^2+b^2}} \quad \text{and sub.}$$

$$\boxed{\vec{r}(s) = \left\langle a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right\rangle}$$

Verify that s is arc length parameter!

$$\frac{d\vec{r}(s)}{ds} = \left\langle -\frac{a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{\sqrt{a^2+b^2}} \right\rangle$$

$$\left| \frac{d\vec{r}(s)}{ds} \right| = \sqrt{\frac{a^2}{a^2+b^2}\sin^2(\ ) + \frac{a^2}{a^2+b^2}\cos^2(\ ) + \frac{b^2}{a^2+b^2}}$$

$$= \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = 1$$

Ex 2 (return)

$$\boxed{\vec{r}(t) = \langle \cos(t^2), \sin(t^2), 0 \rangle} \quad 0 \le t \le \sqrt{\pi}$$

$$\boxed{s = t^2}$$

$$t = \sqrt{s}$$

$$0 \le t \le \sqrt{\pi}$$

$$\downarrow$$

$$0 \le s \le \pi,$$

$$\boxed{\vec{r}(s) = \langle \cos(s), \sin(s), 0 \rangle \quad 0 \le s \le \pi}$$

Note: It is standard to use the parameter s only if the parameter is arc length. For all other cases we use t or u ...

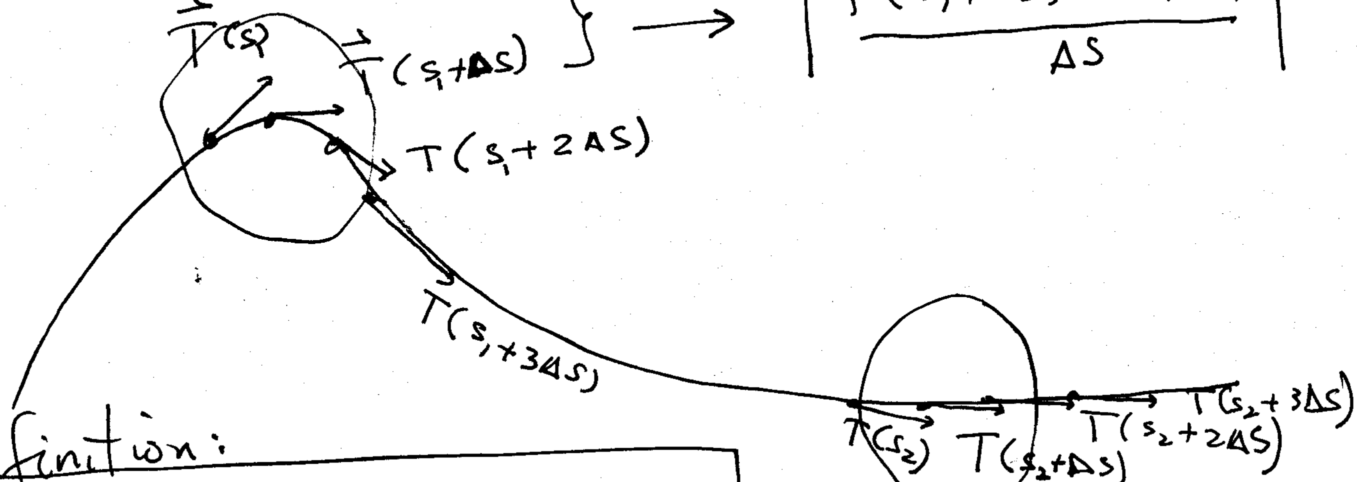
$$\boxed{\vec{r}(s) = \ldots} \qquad \left| \frac{d\vec{r}(s)}{ds} \right| = 1$$

$$\vec{r}(t) = \qquad \left| \frac{d\vec{r}(t)}{dt} \right| \text{ may not be 1}$$

Curvature?

consider a smooth curve, $\vec{r}(s)$, parameterized by arc length.

unit tangent vector $= \vec{T}(s) = \dfrac{d\vec{r}(s)}{ds}$



$$\longrightarrow \left|\frac{T(s_1+\Delta s) - T(s_1)}{\Delta s}\right|$$

definition:

$$\boxed{\kappa(s) = \text{curvature} = \left|\frac{d\vec{T}(s)}{ds}\right|}$$

$$\left|\frac{T(s_2+\Delta s) - T(s_2)}{\Delta s}\right|$$

Ex 3

$$\vec{r}(s) = \left\langle a\cos\left(\tfrac{1}{a}s\right), a\sin\left(\tfrac{1}{a}s\right), 0\right\rangle; \ a>0$$

(circle of radius a parameterized by arc length)

$$\vec{T}(s) = \frac{d\vec{r}(s)}{ds} = \left\langle a \cdot -\frac{1}{a}\sin\left(\tfrac{1}{a}s\right),\ a \cdot \tfrac{1}{a}\cos\left(\tfrac{1}{a}s\right), 0 \right\rangle$$

$$\boxed{\vec{T}(s) = \left\langle -\sin\left(\tfrac{1}{a}s\right),\ \cos\left(\tfrac{1}{a}s\right), 0 \right\rangle}$$

$$\text{curvature} = \kappa(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\|$$

$$= \left\| \left\langle -\tfrac{1}{a}\cos\left(\tfrac{1}{a}s\right),\ -\tfrac{1}{a}\sin\left(\tfrac{1}{a}s\right), 0 \right\rangle \right\|$$

$$= \left(\frac{1}{a^2}\cos^2\left(\tfrac{1}{a}s\right) + \frac{1}{a^2}\sin^2\left(\tfrac{1}{a}s\right) \right)^{1/2}$$

$$\boxed{\kappa(s) = \frac{1}{a}}$$ curvature of a circle.

Note: It is <u>not</u> correct to write:

$$\kappa(t) = \left| \frac{d\vec{T}}{dt} \right| \qquad \vec{r}(t)$$

when t is <u>not</u> arc length.

If $\vec{r}(t)$, t <u>not</u> arc length parameter use the following to calculate curvature.

1) $$\kappa(t) = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$ ✓

or better

2) $$\kappa(t) = \frac{|\vec{a}\times\vec{v}|}{|\vec{v}|^3} = \frac{|\vec{r}''(t)\times\vec{r}'(t)|}{|\vec{r}'(t)|^3}$$

<u>Ex 4</u> <u>Return to the Helix</u>

$\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ $\underline{a>0, b>0 \quad t\geq 0}$

Find the <u>curvature function, $\kappa(t)$</u>.

$\vec{v} = \vec{r}'(t) = \langle -a\sin t, a\cos t, b\rangle$

$\vec{a} = \vec{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle$

$|\vec{v}| = \sqrt{a^2+b^2}$

$$\vec{a} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -a\cos t & -a\sin t & 0 \\ -a\sin t & a\cos t & b \end{vmatrix}$$

$$\vec{a} \times \vec{v} = \langle -ab\sin t,\ ab\cos t,\ -a^2\cos^2 t - a^2\sin^2 t \rangle$$

$$\vec{a} \times \vec{v} = \langle -ab\sin t,\ ab\cos t,\ -a^2 \rangle$$

$$|\vec{a} \times \vec{v}| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$$

$$|\vec{a} \times \vec{v}| = \sqrt{a^2b^2 + a^4} = a\sqrt{b^2 + a^2}$$

$$\kappa(t) = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3} = \frac{a\sqrt{b^2+a^2}}{\left(\sqrt{a^2+b^2}\right)^3} = \frac{a}{a^2+b^2}$$

$$\boxed{\kappa(t) = \frac{a}{a^2+b^2}}$$