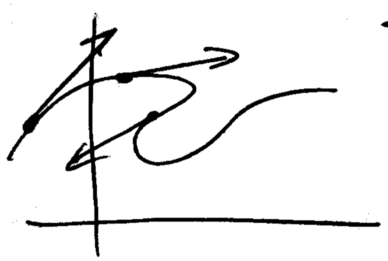
Quiz 3 - 11.6

Exam 1 - Tues 2/21 - Chapter 11.
You may bring one 3x5 card w/ formulas.
No calculators.

#41 (a) False.



$\vec{r}(t)$.
Even if $|\vec{r}\,'(t)|$ is constant
$\vec{r}\,'(t)$ need not be.

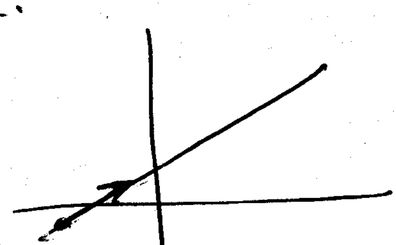
(b) $\vec{r}(t) = \langle \cos t, \sin t \rangle$ $\quad \vec{R}(t) = \langle \cos t^2, \sin t^2 \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$ $\quad \vec{R}\,'(t) = \langle -2t \sin t^2, 2t \cos t^2 \rangle$

$|\vec{r}\,'(t)| = (\sin^2 t + \cos^2 t)^{1/2}$
$= 1$

$$|\vec{R}\,'(t)| = \left(4t^2 \sin^2(t^2) + 4t^2 \cos^2(t^2)\right)^{1/2}$$
$$= \left(4t^2 \underbrace{(\sin^2(t^2) + \cos^2(t^2))}_{1}\right)^{1/2}$$
$$= 2t$$

(c) False.



$|\vec{r}\,'(t)|$ need not be constant.

$\vec{r}\,'(t)$ - const direction
means $\underbrace{\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\text{means motion is on a line}} = \text{const.}$

unit tangent vector.

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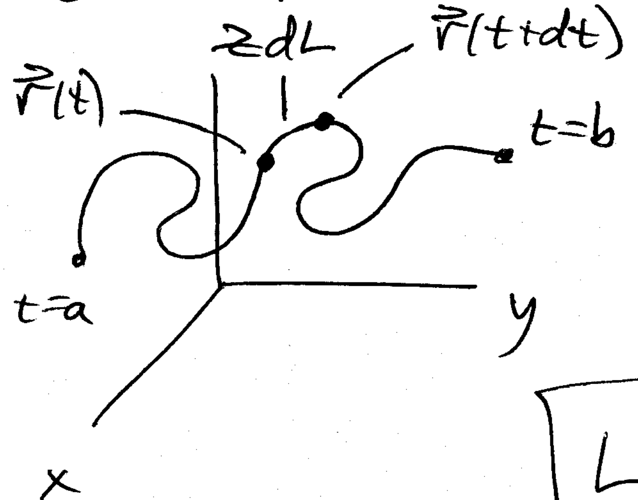
11.8 Arc Length

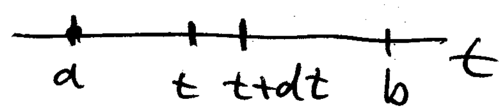
<u>Problem</u>: Given $\vec{r}(t)$, $a \le t \le b$ find length of curve traced out by $\vec{r}(t)$.



$$dL = |\vec{r}(t+dt) - \vec{r}(t)|$$

$$= |\vec{r}\,'(t)|\,dt$$

$$\boxed{L = \int_a^b dL = \int_a^b |\vec{r}\,'(t)|\,dt}$$



<u>Idea</u>: 1. Partition $[a,b]$ into infinitesimal subintervals of length dt.

2. Find dL = arclength element corresp. to the segment of length dt.

3. "Add up" all the dL with an integral.

$$|\vec{r}\,'(t)| = \lim_{\Delta t \to 0} \frac{|\vec{r}(t+\Delta t) - \vec{r}(t)|}{\Delta t}$$

Replace Δt by dt, then $|\vec{r}\,'(t)| \approx \dfrac{|\vec{r}(t+dt) - \vec{r}(t)|}{dt}$

Think of dt as infinitesimally small, we have differential formula, $|\vec{r}\,'(t)|\,dt = |\vec{r}(t+dt) - \vec{r}(t)|$

Think of $\vec{r}(t)$ as describing motion:

$|\vec{r}\,'(t)|$ = speed at time t.

$|\vec{r}\,'(t)|\,dt$ = (speed) × (time interval)

= dist travelled betw t and t+dt.

$= dL$

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eg #10) $\vec{r}(t) = \langle \cos t + \sin t, \cos t - \sin t \rangle$

$0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -\sin t + \cos t, -\sin t - \cos t \rangle$

$|\vec{r}\,'(t)| = ((\cos t - \sin t)^2 + (\cos t + \sin t)^2)^{1/2}$

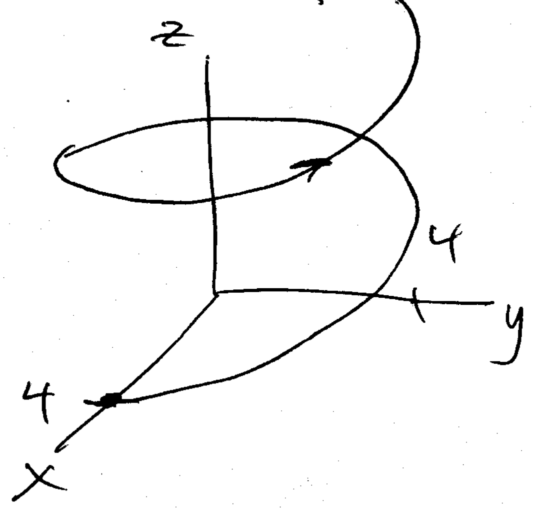
$= (\cos^2 t - \cancel{2 \sin t \cos t} + \sin^2 t$

$+ \cos^2 t + \cancel{2 \cos t \sin t} + \sin^2 t)^{1/2}$

$= \sqrt{2}$

$L = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2} \cdot 2\pi = \underline{\underline{2\sqrt{2}\,\pi}}$

#12) $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle \quad 0 \le t \le 6\pi$



$\vec{r}'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$

$|\vec{r}'(t)| = (16\sin^2 t + 16\cos^2 t + 9)^{1/2}$

$= (16+9)^{1/2} = 5.$

$L = \int_0^{6\pi} 5\,dt = 30\pi$ //

#20) $\vec{r}(t) = \langle t^2, 2t^2, t^3 \rangle \quad 1 \le t \le 2$

$\vec{r}'(t) = \langle 2t, 4t, 3t^2 \rangle$

$|\vec{r}'(t)| = (4t^2 + 16t^2 + 9t^4)^{1/2}$

$= (20t^2 + 9t^4)^{1/2}$

$= t(20 + 9t^2)^{1/2}$

$$L = \int_1^2 |\vec{r}\,'(t)|\,dt = \frac{1}{18}\int_1^2 18t\,(20+9t^2)^{1/2}\,dt$$

$u = 20 + 9t^2$

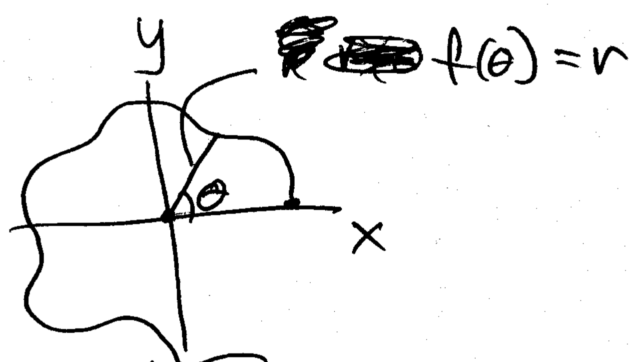
$du = 18t\,dt$

$t = 1 \quad u = 29$

$t = 2 \quad u = 56$

$$= \int_{29}^{56} u^{1/2}\,du = \cdots$$

#28) $r = f(\theta)$



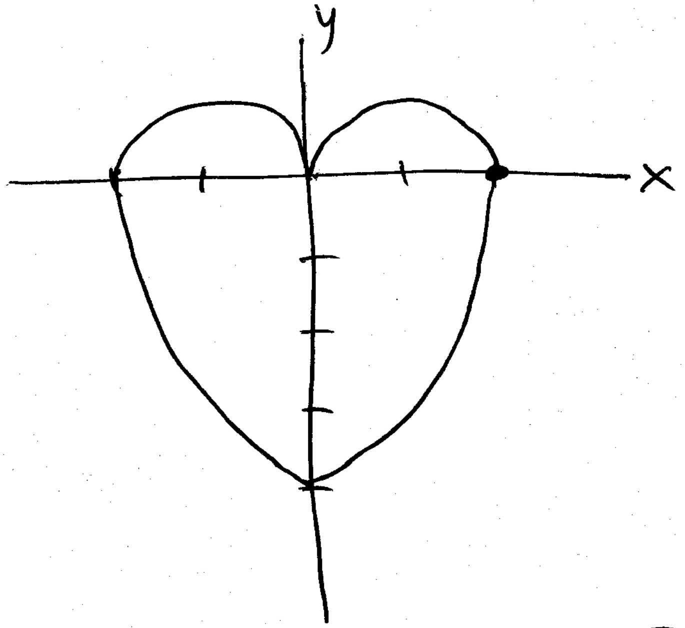
Idea: Realize this as parametrized curve with θ as parameter.

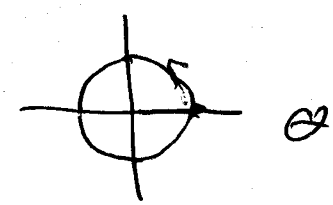
$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$\vec{r}(\theta) = \langle f(\theta)\cos\theta,\ f(\theta)\sin\theta \rangle$

$r = 2 - 2\sin\theta$





$\theta = 0 \quad r = 2$

$\theta = \frac{\pi}{2} \quad r = 0$

$\theta = \pi \quad r = 2$

$\theta = \frac{3\pi}{2} \quad r = 4$

$\theta = 2\pi \quad r = 2$

$x = r\cos\theta = (2-2\sin\theta)\cos\theta = 2(\cos\theta - \sin\theta\cos\theta)$

$y = r\sin\theta = (2-2\sin\theta)\sin\theta = 2(\sin\theta - \sin^2\theta)$

$dL = \left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{1/2}$ $\quad \frac{dx}{d\theta} = 2(-\sin\theta + \sin^2\theta - \cos^2\theta)$

$= 2(-\sin\theta - \cos 2\theta)$

$\frac{dy}{d\theta} = 2(\cos\theta - 2\sin\theta\cos\theta) = 2(\cos\theta - \sin 2\theta)$

From this we get: $dL = (8(1-\sin\theta))^{1/2}$

$= 2\sqrt{2}(1-\sin\theta)^{1/2}$.

$L = \int_0^{2\pi} 2\sqrt{2}(1-\sin\theta)^{1/2}\, d\theta.$

11.9 Curvature + Normal Vectors.

A. Arc length parameter.

Know $\vec{r}(t)$, $\vec{R}(s)$ can have same graph but different dynamics.

Q: Is there a way to reparametrize any curve $\vec{r}(t)$ to $\vec{R}(s)$ so that $|\vec{R}'(s)| = 1$. More later.