11.4 Cross Product

A. How to compute.

1. Determinant of a matrix

2x2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3x3 — Method of cofactors.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

eg

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (1)\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - (2)\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + (3)\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (1)(2-(-1)) - (2)(4-0) + (3)(2-0)$$

$$= 3 - 8 + 6 = 1.$$

2. Cross product.

$\vec{u} = \langle u_1, u_2, u_3 \rangle \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

eg. $\vec{u} = 2\vec{i} + \vec{j} + \vec{k}$

$\vec{v} = -4\vec{i} + 3\vec{j} + \vec{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= \vec{i}(1-3) - \vec{j}(2+4) + \vec{k}(6+4)$$

$$= -2\vec{i} - 6\vec{j} + 10\vec{k}.$$

B. Properties of Cross Product.

1. Algebraic properties

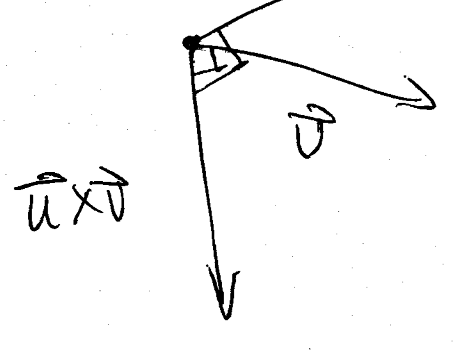
   See Thm 11.4

   not commutative $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

   not associative

   $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ in general.

2. Geometric properties



a) $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$

b) direction given by right-hand-rule.

e.g. verify that $\vec{u} = \langle 2,1,1 \rangle$ $\vec{v} = \langle -4,3,1 \rangle$

$\vec{u} \times \vec{v}$ perpendicular to $\vec{u}$ and $\vec{v}$

$\vec{u} \times \vec{v} = \langle -2, -6, 10 \rangle$

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = \langle -2, -6, 10 \rangle \cdot \langle 2,1,1 \rangle$$
$$= -4 - 6 + 10 = 0$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle -2, -6, 10 \rangle \cdot \langle -4,3,1 \rangle = 8 - 18 + 10 = 0.$$

c) length of $\vec{u} \times \vec{v}$.

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$

θ = angle between $\vec{u}$ and $\vec{v}$ $(0 \le \theta \le \pi)$

if $\vec{u}$ and $\vec{v}$ are parallel, $\vec{u} \times \vec{v} = \vec{0}$.