MATH 213 – 9 MAY 2006 – FINAL EXAM

Answer each of the following questions. Show all work, as partial credit may be given. This exam will be counted out of a total of 100 points.

1. (4 pts. each) Consider the function $f(x, y) = x^2y^3 + 2x^4y$.

   (a) Find $\nabla f(x, y)$.

   (b) Find the direction of greatest increase of f at the point $(1, 2)$ and find the rate of change of f in that direction.

   (c) Find the rate of change of f at the point $(1, 2)$ in the direction $\mathbf{i} - 3\mathbf{j}$.

   (d) Find the differential df of the function $f(x, y)$.

   (e) Find the linearization of $f(x, y)$ at the point $(1, 2)$.

   (f) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 2, 12)$.

2. (8 pts. each) Consider the function $f(x, y) = 2x^3 + y^3 - 3x^2 - 12x - 3y$.

   (a) Find all critical points of $f(x, y)$. (Hint: There are four.)

   (b) Identify each of the critical points you found in part (a) as a local maximum, local minimum, or saddle point.

3. (6 pts. each) Consider the iterated double integral $\int_0^1 \int_0^{y^2} 4xy\,dx\,dy$.

   (a) Evaluate the double integral given above.

   (b) Write an iterated double integral in the order $dy\,dx$ equivalent to the above integral. DO NOT EVALUATE.

   (c) Find the Jacobian of the transformation $x = u^2$, $y = v$.

   (d) Use the transformation $x = u^2$, $y = v$ to transform the double integral given at the beginning of this problem into an iterated integral in the variables u and v. DO NOT EVALUATE. (Hint: You may assume that $u > 0$ and $v > 0$.)

4. (8 pts. each)

(a) Calculate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x,y) = xy + x^2$ and where C is the graph of the function $y = x^3 + x$ from $(-2,-10)$ to $(1,2)$. (Hint: Stop and think! This is easier than it looks.)

(b) Show that the vector field $\mathbf{F}(x,y) = (x^2+y^2)\,\mathbf{i} + (2xy+3y^2)\,\mathbf{j}$ is conservative by finding a function $f(x,y)$ such that $\mathbf{F}(x,y) = \nabla f(x,y)$.

5. (8 pts. each) Consider the planar vector field given by $\mathbf{F} = xy\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j}$ and the closed curve C consisting of the graph of $y = x^2$ from $(0,0)$ to $(1,1)$ and the graph of $y = x$ from $(1,1)$ to $(0,0)$. **Use Green's Theorem** to find

(a) the counterclockwise circulation of $\mathbf{F}$ around C, that is, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and

(b) the outward flux of $\mathbf{F}$ through C, that is, $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

6. (4 pts. each) Let $\mathbf{F}(x,y) = (xz + y^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + z^3\,\mathbf{k}$.

(a) Find div($\mathbf{F}$). (Hint: div($\mathbf{F}$) = $\nabla \cdot \mathbf{F}$)

(b) Find curl($\mathbf{F}$). (Hint: curl($\mathbf{F}$) = $\nabla \times \mathbf{F}$)

(c) Find div(curl($\mathbf{F}$)). (Hint: Stop and think! This is easier than it looks.)