MATH 213 – FINAL EXAM – SOLUTIONS

1. $R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$

$dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2$

$\frac{\partial R}{\partial R_1} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-2}\left(\frac{-1}{R_1^2}\right) = \frac{1}{R_1^2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-2}$

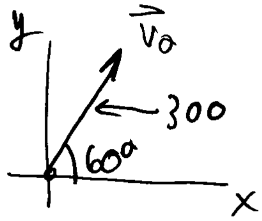
$\frac{\partial R}{\partial R_2} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-2}\left(\frac{-1}{R_2^2}\right) = \frac{1}{R_2^2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-2}$

$R_1 = 300,\ dR_1 = 15,\ R_2 = 600,\ dR_2 = 6$

$\therefore dR = \frac{1}{(300)^2}\left(\frac{1}{300} + \frac{1}{600}\right)^{-2}(15) + \frac{1}{(600)^2}\left(\frac{1}{300} + \frac{1}{600}\right)^{-2}(6)$

$= \frac{15}{(300)^2}\cdot(200)^2 + \frac{6}{(600)^2}(200)^2$

$= 15\cdot\frac{4}{9} + 6\cdot\frac{1}{9} = \frac{66}{9} = \frac{22}{3} \approx 7.33\ \Omega$ //

2. 

(a) $\vec{v}_0 = 300\cos(60°)\vec{i} + 300\sin(60°)\vec{j}$

$= 150\vec{i} + 150\sqrt{3}\,\vec{j}$

(b) $\vec{r}_0 = \vec{0}$ $\therefore$ $\vec{r}(t) = -16t^2\vec{j} + (150\vec{i} + 150\sqrt{3}\vec{j})t$

$= 150t\,\vec{i} + (150\sqrt{3}t - 16t^2)\vec{j}$

(c) $150\sqrt{3}t - 16t^2 = 0 \rightarrow t = 0,\ t = \frac{150\sqrt{3}}{16} \approx 16.2$ sec. //

$150(16.2) \approx 2435.7$ feet //

3. $f(x,y) = x^2 + 2y^2 + x^2y + 4$

(a) $f_x = 2x + 2xy$ $\qquad$ $2x + 2xy = 0$

$2x(1+y) = 0$

$f_y = 4y + x^2$ $\qquad$ $x = 0 \quad y = -1$

$x = 0 \longrightarrow 4y = 0 \quad \therefore y = 0$ $\qquad$ $(0,0)$

$y = -1 \longrightarrow -4 + x^2 = 0 \quad \therefore x = \pm 2$ $\quad (2,-1), (-2,-1)$

(b) $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 2 + 2y$

$f_{yy} = 4$

$f_{xy} = 2x$

$f_{xx}(0,0) = 2$

$f_{yy}(0,0) = 2$

$f_{xy}(0,0) = 0$

$\therefore D(0,0) = 4 > 0$

$f_{xx}(0,0) > 0$

$\therefore (0,0)$ local min

$f_{xx}(2,-1) = 0$

$f_{yy}(2,-1) = 4$

$f_{xy}(2,-1) = 4$

$D(2,-1) = -(4)^2$

$= -16 < 0$

$(2,-1)$ saddle

$f_{xx}(-2,-1) = 0$

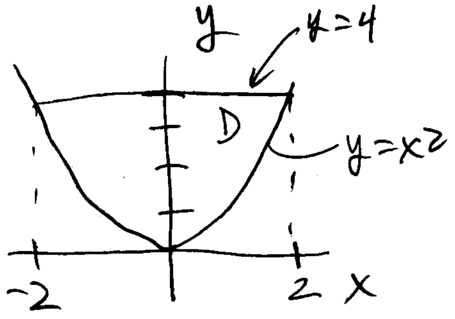
$f_{yy}(-2,-1) = 4$

$f_{xy}(-2,-1) = -4$

$D(-\sqrt{2},-1) = -(-4)^2 = -16 < 0$

$(-2,-1)$ saddle

4.



(a) $\iint_D (x^2+xy)\,dA$

$= \int_{-2}^{2}\int_{x^2}^{4} (x^2+xy)\,dy\,dx$ //

$= \int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}} (x^2+xy)\,dx\,dy$ //

(b) $\int_{-2}^{2}\int_{x^2}^{4} (x^2+xy)\,dy\,dx$

$= \int_{-2}^{2} x^2y + \frac{1}{2}xy^2 \Big|_{x^2}^{4}\,dx$

$= \int_{-2}^{2} (4x^2+8x-x^4-\frac{1}{2}x^5)\,dx$

$= \frac{4}{3}x^3+4x^2-\frac{1}{5}x^5-\frac{1}{12}x^6\Big|_{-2}^{2}$

$= \frac{4}{3}(8)+\cancel{4(4)}-\frac{1}{5}(32)-\cancel{\frac{1}{12}(64)}-\frac{4}{3}(-8)-\cancel{4(4)}+\frac{1}{5}(-32)+\cancel{\frac{1}{12}(64)}$

$= \frac{64}{3}-\frac{64}{5} = \frac{128}{15} \approx 8.53$

$\int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} (x^2+xy)\,dx\,dy = \int_0^4 \frac{1}{3}x^3+\frac{1}{2}x^2y\Big|_{-\sqrt{y}}^{\sqrt{y}}\,dy$

$= \int_0^4 \frac{1}{3}y^{3/2}+\frac{1}{2}y^2+\frac{1}{3}y^{3/2}-\frac{1}{2}y^2\,dy = \int_0^4 \frac{2}{3}y^{3/2}\,dy$

$= \frac{2}{3}\cdot\frac{2}{5}\left(y^{5/2}\Big|_0^4\right) = \frac{4}{15}(4)^{5/2} = \frac{4}{15}\cdot 2^5 = \frac{128}{15}$ //

5. $\int_C \vec{F} \cdot d\vec{r}$ $\quad \vec{F} = x^2 \vec{i} + xy \vec{j} - z^2 \vec{k}$

$C: \vec{r}(t) = (1-t)(2\vec{j} + \vec{k}) + t(4\vec{i} + 2\vec{j} + 5\vec{k})$

$= 4t\vec{i} + (\cancel{2-2t}+2t)\vec{j} + (1-t+5t)\vec{k}$

$= 4t\vec{i} + \cancel{(2)}\vec{j} + (1+4t)\vec{k} \quad 0 \le t \le 1$

$\vec{r}'(t) = 4\vec{i} + 4\vec{k}$

$= \int_0^1 \left((4t)^2 \vec{i} + (\cancel{4t)(1+t}) \vec{j} - (1+4t)^2 \vec{k}\right) \cdot (4\vec{i} + 4\vec{k})\, dt$ $\quad 8t$

$= \int_0^1 \left[64t^2 + \cancel{\quad} - 4(1+8t+16t^2)\right] dt$

$= \int_0^1 64t^2 + \cancel{\quad} - 4 - 32t - 64t^2 \, dt$

$= \cancel{\quad} - 4t - 16t^2 \Big|_0^1 = -20$ // $\cancel{\quad}$ //.

6. $\int_C xy^2z^3 \, ds$ $\quad C: \vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t^2\vec{k}$

$0 \le t \le 2\pi$

$= \int_0^{2\pi} (2\cos t)(2\sin t)^2 (t^2)^3 \cdot 2\sqrt{1+t^2}\, dt$

$\vec{r}'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 2t\,\vec{k}$

$|\vec{r}'(t)| = (4\sin^2 t + 4\cos^2 t + 4t^2)^{1/2}$

$= 2(1+t^2)^{1/2}$

$ds = |\vec{r}'(t)|\,dt = 2\sqrt{1+t^2}\,dt$

$= 16 \int_0^{2\pi} (\cos t \sin^2 t) t^6 (1+t^2)^{1/2} dt$

7. (a) $\vec{F} = (8xy + y)\vec{i} + (4x^2 + x + 2y)\vec{j}$

$f_x = 8xy + y$

$f = 4x^2y + xy + g(y)$

$f_y = 4x^2 + x + g'(y) = 4x^2 + x + 2y$

$\therefore g'(y) = 2y \quad \therefore g(y) = y^2$

$\therefore f = 4x^2y + xy + y^2$ //.

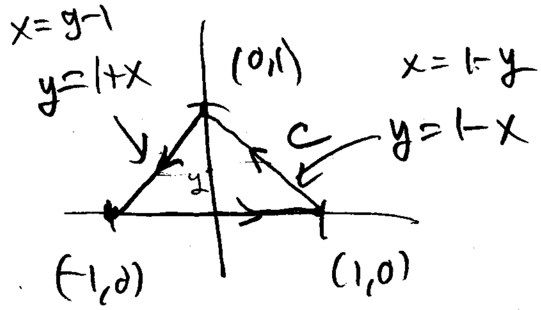
(b) $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^3+z^2 & x^2+y^2+z^2 & xyz \end{vmatrix}$

$= \vec{i}(xz - 2z) - \vec{j}(yz - 2z) + \vec{k}(2x - 3xy^2) \neq 0.$

since $\operatorname{curl}\vec{F} \neq \vec{0}$, $\vec{F}$ is <u>not</u> conservative.

(c) $\operatorname{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = y^3 + 2y + xy$ //.

8.



$$\int_C xy^2\,dx + 2x^2y\,dy = \iint_D (4xy - 2xy)\,dA$$

$$= \iint_D 2xy\,dA = \int_0^1 \int_{y-1}^{1-y} 2x\,y\,dx\,dy$$

$$= \int_0^1 x^2 y \Big|_{y-1}^{1-y} dy = \frac{1}{3}\int_0^1 y(1-y)^2 - y(y-1)^2\,dy = 0$$

$$y(1-y)^2 - y(y-1)^2 = y(1-2y+y^2) - y(y^2-2y+1)$$

$$= \cancel{y} - \cancel{2y^2} + \cancel{y^3} - \cancel{y^3} + \cancel{2y^2} - \cancel{y} = 0$$