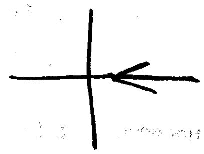
MATH 213 - EXAM 2 - SOLUTIONS

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{3x^2+y^2}$$

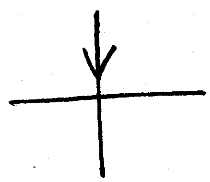
path 1:



$y = 0, x>0$

$$\lim_{x\to 0^+} \frac{x^2}{3x^2} = \lim_{x\to 0^+} \frac{1}{3} = \frac{1}{3}$$

path 2:



$x=0, y>0$

$$\lim_{y\to 0^+} \frac{3y^2}{y^2} = \lim_{x\to 0^+} 3 = 3$$

Different limits along different paths means the limit does not exist.

2. $f(x,y,z) = x^3y + y^3z + z^3x$

(a) $\nabla f(x,y,z) = (3x^2y + z^3)\vec{i} + (3y^2z + x^3)\vec{j} + (3z^2x + y^3)\vec{k}$

(b) $$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3\vec{i}+6\vec{j}-2\vec{k}}{(9+36+4)^{1/2}} = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}$$

$$\nabla f(1,0,2) = 8\vec{i} + \vec{j} + 12\vec{k}$$

$$\begin{aligned} D_{\vec{u}} f(1,0,2) &= \nabla f(1,0,2) \cdot \vec{u} \\ &= (8\vec{i} + \vec{j} + 12\vec{k}) \cdot (\tfrac{3}{7}\vec{i} + \tfrac{6}{7}\vec{j} - \tfrac{2}{7}\vec{k}) \\ &= 6/7 \ // . \end{aligned}$$

(c) MAX RATE OF CHANGE AT $(1,0,2)$ IS

$$\begin{aligned} |\nabla f(1,0,2)| &= |8\vec{i} + \vec{j} + 12\vec{k}| \\ &= (64 + 1 + 144)^{1/2} = (209)^{1/2} \ // \end{aligned}$$

DIRECTION OF MOST RAPID INCREASE IS

$$\frac{\nabla f(1,0,2)}{|\nabla f(1,0,2)|} = \frac{8}{\sqrt{209}}\vec{i} + \frac{1}{\sqrt{209}}\vec{j} + \frac{12}{\sqrt{209}}\vec{k} \ //$$

3. $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial x} = y \qquad \frac{\partial w}{\partial y} = x \qquad \frac{\partial w}{\partial z} = \frac{1}{z}$$

$$\frac{\partial x}{\partial u} = \frac{-v^2}{u^2} \qquad \frac{\partial x}{\partial v} = \frac{2v}{u}$$

$$\frac{\partial y}{\partial u} = 1 \qquad \frac{\partial y}{\partial v} = 1$$

$$\frac{\partial z}{\partial u} = -\sin(u) \qquad \frac{\partial z}{\partial v} = 0$$

$$\therefore \frac{\partial w}{\partial u} = y\left(\frac{-v^2}{u^2}\right) + x(1) + \frac{1}{z}(-\sin(u))$$

$$= -\frac{v^2(u+v)}{u^2} + \frac{v^2}{u} - \frac{\sin(u)}{\cos(u)} \ //$$

$$\frac{\partial w}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \frac{1}{z}(0)$$

$$= \frac{2(u+v)v}{u} + \frac{v^2}{u} \ //$$

4. (a) $f(x,y) = x\sin(xy)$

$f_x = xy\cos(xy) + \sin(xy)$

$f_{xx} = -xy^2\sin(xy) + 2y\cos(xy)$

$f_y = x^2\cos(xy)$

$f_{yy} = -x^3\sin(xy)$

$f_{xy} = f_{yx} = -x^2y\sin(xy) + 2x\cos(xy)$

(b) $f(x,y) = \ln(xy^2) = \ln(x) + 2\ln(y)$

$f_x = \frac{1}{x}$ $\quad f_y = \frac{2}{y}$ $\quad f_{xy} = f_{yx} = 0$

$f_{xx} = \frac{-1}{x^2}$ $\quad f_{yy} = \frac{-2}{y^2}$

5. $f(x,y,z) = \dfrac{yz^2}{x^3}$

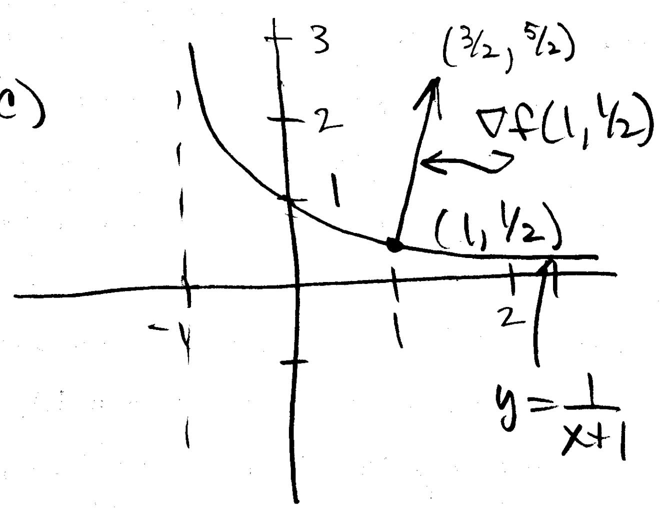
$f_z = \dfrac{2yz}{x^3}$ $\qquad f_{zx} = \dfrac{-6yz}{x^4}$ //

$f_{yz} = f_{zy} = \dfrac{2z}{x^3}$ //

$f_x = \dfrac{-3yz^2}{x^4}$ $\qquad f_{xy} = \dfrac{-3z^2}{x^4}$ //

$f_{zyz} = f_{yzz} = (f_{yz})_z = \dfrac{\partial}{\partial z}\left(\dfrac{2z}{x^3}\right) = \dfrac{2}{x^3}$ //

6. (a) (c)



$xy + y = 1$

$y(x+1) = 1$

$y = \dfrac{1}{x+1}$

(b) $\nabla f(x,y) = y\vec{i} + (x+1)\vec{j}$ $\qquad \nabla f(1, \tfrac{1}{2}) = \tfrac{1}{2}\vec{i} + 2\vec{j}$