MATH 213 - EXAM 1 - SOLUTIONS

1. $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ $\quad \vec{b} = 3\vec{i} + 5\vec{j} - 4\vec{k}$ $\quad \vec{c} = \vec{i} + \vec{j} - 3\vec{k}$

(a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$ $\qquad \vec{a}\cdot\vec{b} = (1)(3) + (2)(5) - (2)(4) = 5$

$$|\vec{a}| = (1+4+4)^{1/2} = 3$$

$$|\vec{b}| = (9+25+16)^{1/2} = 5\sqrt{2}$$

$$\therefore \cos\theta = \frac{5}{15\sqrt{2}} = \frac{1}{3\sqrt{2}} \approx .236$$

(b) $\vec{a} = |\vec{a}|\left(\dfrac{\vec{a}}{|\vec{a}|}\right) = 3\left(\tfrac{1}{3}\vec{i} + \tfrac{2}{3}\vec{j} + \tfrac{2}{3}\vec{k}\right)$

(c) $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 3 & 5 & -4 \end{vmatrix} = -18\vec{i} + 10\vec{j} - \vec{k}$ //

(d) $\text{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a} = \dfrac{5}{9}\vec{a} = \dfrac{5}{9}\vec{i} + \dfrac{10}{9}\vec{j} + \dfrac{10}{9}\vec{k}$ //

(e) $\vec{b} = \text{proj}_{\vec{a}}(\vec{b}) + \left(\vec{b} - \text{proj}_{\vec{a}}(\vec{b})\right)$

$$= \left(\tfrac{5}{9}\vec{i} + \tfrac{10}{9}\vec{j} + \tfrac{10}{9}\vec{k}\right) + \left(\tfrac{22}{9}\vec{i} + \tfrac{35}{9}\vec{j} - \tfrac{46}{9}\vec{k}\right) \quad //$$

(f) Area $= |\vec{a}\times\vec{b}| = (324+100+1)^{1/2} = (425)^{1/2} = 5\sqrt{17}$

≈ 20.6 //

(g) Volume $= |(\vec{a}\times\vec{b})\cdot\vec{c}|$

$$= \left|(-18\vec{i}+10\vec{j}-\vec{k})\cdot(\vec{i}+\vec{j}-3\vec{k})\right|$$

$$= |-18+10+3| = 5 \quad //$$

(h) $x = 6+t$

$y = 5+2t$

$z = -1+2t \quad //$

(i) $(1)(x-6)+2(y-5)+2(z+1)=0$

$x-6+2y-10+2z+2=0$

$x+2y+2z=14 \quad //$

2. $\vec{r}(t) = (v_0\cos\alpha)t\,\vec{i} + (-\frac{1}{2}gt^2 + v_0\sin\alpha\, t)\vec{j}$

$v_0 = 500 \quad \alpha = \frac{\pi}{6} \quad g = 9.8$

$\therefore \vec{r}(t) = 500\cdot\frac{\sqrt{3}}{2}\,t\,\vec{i} + (-4.9t^2 + 500\cdot\frac{1}{2}t)\vec{j}$

$$= 250\sqrt{3}\,t\,\vec{i} + (-4.9t^2+250t)\vec{j} \quad //$$

(a) $\vec{r}'(t) = 250\sqrt{3}\,\vec{i} + (-9.8t+250)\vec{j}$

Rate of change of height occurs when $-9.8t+250=0$

or $t = \frac{250}{9.8} \approx 25.5$

Max height $= -4.9\left(\frac{250}{9.8}\right)^2 + 250\left(\frac{250}{9.8}\right) = 3189\text{ m} \quad //$

(b) Projectile at height zero when

$$-4.9t^2 + 250t = 0$$

$$t(-4.9t + 250) = 0$$

$$\cancel{t=0} \qquad t = \frac{250}{4.9} \approx 51.0 \text{ sec}$$

$$\text{Range} = 250\sqrt{3}\left(\frac{250}{4.9}\right) \approx 22092 \text{ m} \quad //$$

3. $\vec{r}(t) = t\,\vec{i} + \frac{1}{3}t^{3/2}\,\vec{j} + t\,\vec{k}$

(a) $\vec{r}\,'(t) = \vec{i} + \frac{1}{2}t^{1/2}\,\vec{j} + \vec{k}$ //

$\vec{r}\,''(t) = \frac{1}{4}t^{-1/2}\,\vec{j}$ //

$|\vec{r}\,'(t)| = \left(1 + \frac{1}{4}t + 1\right)^{1/2} = \left(2 + \frac{1}{4}t\right)^{1/2}$ //

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{(2+\frac{1}{4}t)^{1/2}}\,\vec{i} + \frac{t^{1/2}}{2(2+\frac{1}{4}t)^{1/2}}\,\vec{j} + \frac{1}{(2+\frac{1}{4}t)^{1/2}}\,\vec{k} \quad //$$

(b) $L = \int_0^4 |\vec{r}\,'(t)|\,dt = \int_0^4 \left(2 + \frac{1}{4}t\right)^{1/2} dt$

$u = 2 + \frac{1}{4}t$

$du = \frac{1}{4}\,dt$

$t = 0 \quad u = 2$

$t = 4 \quad u = 3$

$$= 4\int_2^3 u^{1/2}\,du = 4 \cdot \frac{2}{3} u^{3/2}\Big|_2^3 = \frac{8}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

$$\approx 6.31 \quad //$$

4. $A=(2,4,5)$ $B=(0,0,1)$ $C=(3,-1,2)$

(a) $\vec{n} = \vec{AB} \times \vec{AC} = \langle -2,-4,-4\rangle \times \langle 1,-5,-3\rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -4 & -4 \\ 1 & -5 & -3 \end{vmatrix} = -8\vec{i} - 10\vec{j} + 14\vec{k}$$

point $= (2,4,5)$ (for example)

$$-8(x-2) - 10(y-4) + 14(z-5) = 0$$

$$-8x + 16 - 10y + 40 + 14z - 70 = 0$$

$$-8x - 10y + 14z = 14 \;//$$

(b) direction $= \vec{BC} = \langle 3,-1,1\rangle$

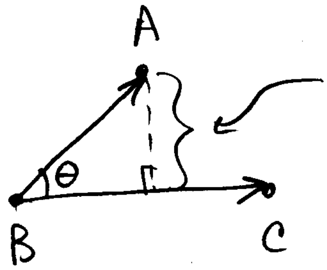
point $= (0,0,1)$ (for example)

$x = 3t$
$y = -t$
$z = 1+t$ //

or choosing point $(3,-1,2)$

$x = 3+3t$
$y = -1-t$
$z = 2+t$ //

(c)



$$\text{dist} = |\vec{BA}| \sin\theta = \frac{|\vec{BA} \times \vec{BC}|}{|\vec{BC}|}$$

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 4 \\ 3 & -1 & 1 \end{vmatrix} = 8\vec{i} + 10\vec{j} - 14\vec{k}$$

$$|\vec{BA} \times \vec{BC}| = (64 + 100 + 196)^{1/2} = \sqrt{360} = 6\sqrt{10}$$

$$|\vec{BC}| = (9 + 1 + 1)^{1/2} = \sqrt{11}$$

$$\therefore \text{distance} = 6\sqrt{\frac{10}{11}} \approx 5.7$$