4.2 Null Spaces, Column Spaces, & Linear Transformations

Definition

The **null space** of an \( m \times n \) matrix \( A \), written as \( \text{Nul} \ A \), is the set of all solutions to the homogeneous equation \( A \mathbf{x} = \mathbf{0} \).

\[
\text{Nul} \ A = \{ \mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A \mathbf{x} = \mathbf{0} \} \quad \text{(set notation)}
\]

**EXAMPLE**  Is \( \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \) in \( \text{Nul} \ A \) where \( A = \begin{bmatrix} 2 & -1 & -1 \\ 4 & -3 & 1 \end{bmatrix} \)?

**Solution:** Determine if \( A \mathbf{w} = \mathbf{0} : \)

\[
\begin{bmatrix} 2 & -1 & -1 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Hence \( \mathbf{w} \) is in \( \text{Nul} \ A \).

**THEOREM 2**

The null space of an \( m \times n \) matrix \( A \) is a subspace of \( \mathbb{R}^n \).

Equivalently, the set of all solutions to a system \( A \mathbf{x} = \mathbf{0} \) of \( m \) homogeneous linear equations in \( n \) unknowns is a subspace of \( \mathbb{R}^n \).

**Proof:** \( \text{Nul} \ A \) is a subset of \( \mathbb{R}^n \) since \( A \) has \( n \) columns. Must verify properties a, b and c of the definition of a subspace.

**Property (a)** Show that \( \mathbf{0} \) is in \( \text{Nul} \ A \). Since ________, \( \mathbf{0} \) is in ________.

**Property (b)** If \( \mathbf{u} \) and \( \mathbf{v} \) are in \( \text{Nul} \ A \) and show that \( \mathbf{u} + \mathbf{v} \) in \( \text{Nul} \ A \). Since \( \mathbf{u} \) and \( \mathbf{v} \) are in \( \text{Nul} \ A \),

\[
\text{________ and ________}.
\]

Therefore

\[
A(\mathbf{u} + \mathbf{v}) = ______ + ______ = ___ + ___ = ___.
\]

**Property (c)** If \( \mathbf{u} \) is in \( \text{Nul} \ A \) and \( c \) is a scalar and show \( c \mathbf{u} \) in \( \text{Nul} \ A \). Since \( \mathbf{u} \) is in \( \text{Nul} \ A \).

\[
A(c \mathbf{u}) = ____A(\mathbf{u}) = c \mathbf{0} = \mathbf{0}.
\]

Since properties a, b and c hold, \( A \) is a subspace of \( \mathbb{R}^n \).
EXAMPLE Find an explicit description of \( \text{Nul } A \) where

\[
A = \begin{bmatrix}
3 & 6 & 6 & 3 & 9 \\
6 & 12 & 13 & 0 & 3
\end{bmatrix}
\]

What do we mean by an \textbf{explicit description of \( \text{Nul } A \)}? 

\textit{Note: Our definition of \( \text{Nul } A \) is called an \textit{implicit description} since we do not list or describe the actual members of the set. By solving \( Ax = 0 \), we will however obtain a specific description of the vectors which we call an \textit{explicit description} of \( \text{Nul } A \).

Solution: Row reduce \( Ax = 0 \) to get:

\[
\begin{bmatrix}
1 & 2 & 0 & 13 & 33 & 0 \\
0 & 0 & 1 & -6 & -15 & 0
\end{bmatrix}
\]

\[
x_1 = -2x_2 - 13x_4 - 33x_5
\]

\[
x_3 = 6x_4 + 15x_5
\]

\( x_2, x_4 \) and \( x_5 \) are free

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= \begin{bmatrix}
-2x_2 - 13x_4 - 33x_5 \\
x_2 \\
6x_4 + 15x_5 \\
x_4 \\
x_5
\end{bmatrix}
= x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
+ x_4 \begin{bmatrix}
-13 \\
0 \\
6 \\
1 \\
0
\end{bmatrix}
+ x_5 \begin{bmatrix}
-33 \\
0 \\
15 \\
0 \\
1
\end{bmatrix}
\]

\[
= x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}
\]

Then

\[
\text{Nul } A = \text{span} \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \}
\]

\textbf{Observations:}

1. Spanning set of \( \text{Nul } A \), found using the method in the last example, is automatically linearly independent:

\[
x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-13 \\
0 \\
6 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
-33 \\
0 \\
15 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Then

\[
x_2 = _____ 
\]

\[
x_4 = _____ 
\]

\[
x_5 = _____ 
\]

2. The number of vectors in the spanning set for \( \text{Nul } A \) equals the number of free variables in \( Ax = 0 \).
Definition

The **column space** of an \( m \times n \) matrix \( A \) is the set of all linear combinations of the columns of \( A \).

*Notation:* \( \text{Col} \ A \) is short for the column space of \( A \).

If \( A = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix} \), then

\[
\text{Col} \ A = \text{Span} \{ a_1, \ldots, a_n \}
\]

**THEOREM 3**

The column space of an \( m \times n \) matrix \( A \) is a subspace of \( \mathbb{R}^m \).

(Why? Reread Theorem 1, page 216.)

Suppose \( A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \) and \( b = A \mathbf{x} \). Then \( b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \)

and this is equivalent to stating that \( b \) is in \( \text{Span} \{ a_1, \ldots, a_n \} \). Therefore

\[
\text{Col} \ A = \left\{ \mathbf{b} : \mathbf{b} = A \mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n \right\}
\]

**EXAMPLE**

Find a matrix \( A \) such that \( W = \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} \) is \( \text{Col} \ A \). Once you find \( A \), determine if \( \text{Col} \ A \) is all of \( \mathbb{R}^3 \).

*Solution:*

\[
\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

Therefore \( A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

Since \( A \) does not have a pivot in every row, the columns of \( A \) do not \( \text{Span} \) all of \( \mathbb{R}^3 \) which means \( \text{Col} \ A \) is not all of \( \mathbb{R}^3 \).

By Theorem 4 (Chapter 1),

\[
\text{The column space of an } m \times n \text{ matrix } A \text{ is all of } \mathbb{R}^m \text{ if and only if the equation } A \mathbf{x} = \mathbf{b} \text{ has a solution for each } \mathbf{b} \text{ in } \mathbb{R}^m.
\]
The Contrast Between \text{Nul} \ A \text{ and } \text{Col} \ A

\text{EXAMPLE} \quad \text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}.

(a) If the column space of \ A is a subspace of \mathbb{R}^k, then \( k = \) ______.

(b) If the null space of \ A is a subspace of \mathbb{R}^k, then what is \( k = \) ______. (Recall that \( x \) is in \text{Nul} \ A if \( A \cdot x = 0 \).)

(c) Find a nonzero vector in \text{Col} \ A. \text{ (There are infinitely many possibilities.)}

\( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \text{vector} \end{bmatrix} \)

(d) Find a nonzero vector in \text{Nul} \ A. \text{ Solve } A \cdot x = 0 \text{ and pick one solution.}

\[ \begin{bmatrix} A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\( x_1 = -2x_2 \) \quad \text{\( x_2 \) is free} \quad \text{\( x_3 = 0 \)}

Let \( x_2 = \) ____ and then

\( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \text{vector} \end{bmatrix} \)
Contrast Between Nul $A$ and Col $A$ where $A$ is $m \times n$

1. Nul $A$ is a subspace of $\mathbb{R}^n$
2. Nul $A$ is implicitly defined; i.e., you must use the condition $A\mathbf{x} = \mathbf{0}$ to actually find Nul $A$
3. It takes time to find vectors in Nul $A$ since you must row reduce $[A \ 0]$ first.
4. There is no obvious relationship between Nul $A$ and entries of $A$.
5. A typical vector $\mathbf{v}$ in Nul $A$ has the property $A\mathbf{x} = \mathbf{0}$.
6. Given a specific vector $\mathbf{v}$, it is easy to determine if $\mathbf{v}$ is in Nul $A$. Just compute $A\mathbf{v}$.
7. Nul $A = \{0\}$ if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

1. Col $A$ is a subspace of $\mathbb{R}^m$
2. Col $A$ is explicitly defined; i.e., you are told how to build specific vectors in Col $A$.
3. It is easy to find vectors in Col $A$ by observing the columns of $A$ and by forming linear combinations of the columns of $A$.
4. There is an obvious relationship between Col $A$ and entries of $A$ since each column of $A$ is in Col $A$.
5. A typical vector $\mathbf{v}$ in Col $A$ has the property $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it takes time to determine if $\mathbf{v}$ is in Col $A$ since you must row reduce $[A \ \mathbf{v}]$ first.
7. Col $A = \mathbb{R}^m$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

Review

DEFINITION

A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.

b. For each $\mathbf{u}$ and $\mathbf{v}$ are in $H$, $\mathbf{u} + \mathbf{v}$ is in $H$. (In this case we say $H$ is closed under vector addition.)

c. For each $\mathbf{u}$ in $H$ and each scalar $c$, $c\mathbf{u}$ is in $H$. (In this case we say $H$ is closed under scalar multiplication.)

If the subset $H$ satisfies these three properties, then $H$ itself is a vector space.

THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space $V$, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of $V$.

The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$.

The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$. 
EXAMPLE  Determine whether each of the following is a vector space or provide a specific example to the contrary.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$  
(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y = 0, y + z = 0 \right\}$

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x - 3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

Solutions: (a) Since $\begin{bmatrix} \cdot & \cdot \end{bmatrix}$ is not in $H$, $H$ is not a vector space.

(b) Rewrite $x - y = 0, y + z = 0$ as $x \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} + y \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} + z \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or as $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $V = \text{Nul } A$ where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\text{Nul } A$ is a subspace of $\mathbb{R}^2$, $V$ is a vector space.

(b) Since $\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$, $S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$ and is therefore a vector space by Theorem 1.
Kernal and Range of a Linear Transformation

Definition

A transformation \(T\) from a vector space \(V\) into a vector space \(W\) is a rule that assigns to each vector \(x\) in \(V\) a unique vector \(T(x)\) in \(W\), such that

- i. \(T(u + v) = T(u) + T(v)\) for all \(u, v\) in \(V\) and
- ii. \(T(cu) = cT(u)\) for all \(u\) in \(V\) and all scalars \(c\).

The kernal of \(T\) is the set of all vectors \(u\) in \(V\) such that \(T(u) = 0\). The range of \(T\) is the set of all vectors of the form \(T(u)\) where \(u\) is in \(V\).

EXAMPLE

Let \(P_2\) be the vector space of all polynomials of degree two or less and define the transformation \(T : P_2 \to \mathbb{R}^2\) such that \(T(p) = \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix}\).

(a) Compute \(T(p)\) if \(p(t) = 2t^2 + 3t + 1\).
(b) Verify that property (i) of a linear transformation holds here.

Solution: (a) \(p'(t) = 4t + 3\) and \(p''(t) = 4\). Then \(T(p) = \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix}\).

(b) Suppose \(p\) and \(q\) are polynomials of degree two or less. Then

\[
T(p + q) = \begin{bmatrix} (p + q)(0) \\ (p + q)''(0) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p''(0) + q''(0) \end{bmatrix} = \begin{bmatrix} p(0) \\ p''(0) \end{bmatrix} + \begin{bmatrix} q(0) \\ q''(0) \end{bmatrix} = T(p) + T(q).
\]