4.2 Null Spaces, Column Spaces, & Linear Transformations

Definition

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

EXAMPLE Is $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ in Nul A where $A = \begin{bmatrix} 2 & -1 & -1 \\ 4 & -3 & 1 \end{bmatrix}$?

Solution: Determine if $A\mathbf{w} = \mathbf{0}$:

$$\begin{bmatrix} 2 & -1 & -1 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $\mathbf{w}$ is in Nul A.

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since _____, $\mathbf{0}$ is in _____.

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A and show that $\mathbf{u} + \mathbf{v}$ in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

_____ and _____.

Therefore

$$A(\mathbf{u} + \mathbf{v}) = \_\_\_\_\_ + \_\_\_\_\_ = \_\_\_ + \_\_\_ = \_\_\_.$$

Property (c) If $\mathbf{u}$ is in Nul A and c is a scalar and show $c\mathbf{u}$ in Nul A. Since $\mathbf{u}$ is in Nul A.

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

EXAMPLE Find an explicit description of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$

What do we mean by an ***explicit description of*** Nul A?

Note: Our definition of Nul A is called an <u>implicit description</u> since we do not list or describe the actual members of the set. By solving $Ax = 0$, we will however obtain a specific description of the vectors which we call an <u>explicit description</u> of Nul A.

Solution: Row reduce $A\mathbf{x} = \mathbf{0}$ to get:

$$\begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$x_1 = -2x_2 - 13x_4 - 33x_5$

$x_3 = 6x_4 + 15x_5$

x_2, x_4 and x_5 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then

$x_2 =$ _____ $x_4 =$ _____ $x_5 =$ _____

2. The number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

Definition

The **column space** of an $m \times n$ matrix A is the set of all linear combinations of the columns of A.

Notation: Col A is short for the column space of A.

If $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}}$$

THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

(Why? Reread Theorem 1, page 216.)

Suppose $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{b} = A\mathbf{x}$. Then $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$

and this is equivalent to stating that **b** is in Span$\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

EXAMPLE Find a matrix A such that $W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} \right\}$ is Col A. Once you find A, determine if Col A is all of $\mathbf{R}^3$.

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{Therefore } A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

Since A does not have a pivot in every row, the columns of A do not Span all of $\mathbf{R}^3$ which means Col A is not all of $\mathbf{R}^3$

By Theorem 4 (Chapter 1),

The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each **b** in $\mathbf{R}^m$.

The Contrast Between Nul A and Col A

EXAMPLE Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) If the column space of A is a subspace of $\mathbf{R}^k$, then $k =$____.

(b) If the null space of A is a subspace of $\mathbf{R}^k$, then what is k=____. (Recall that $\mathbf{x}$ is in Nul A if $A\mathbf{x} = \mathbf{0}$.)

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\_\_\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} \_\_\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2x_2 \\ x_2 &\text{ is free} \\ x_3 &= 0 \end{aligned}$$

Let $x_2 =$ ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Contrast Between Nul A and Col A where A is m $\times$ n

1. Nul A is a subspace of $\mathbf{R}^n$	**1.** Col A is a subspace of $\mathbf{R}^m$
2. Nul A is implicitly defined; i.e., you must use the condition $A\mathbf{x} = \mathbf{0}$ to actually find Nul A	**2.** Col A is explicitly defined; i.e., you are told how to build specific vectors in Col A.
3. It takes time to find vectors in Nul A since you must row reduce $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ first.	**3.** It is easy to find vectors in Col A by observing the columns of A and by forming linear combinations of the columns of A.
4. There is no obvious relationship between Nul A and entries of A.	**4.** There is an obvious relationship between Col A and entries of A since each column of A is in Col A.
5. A typical vector **v** in Nul A has the property $A\mathbf{x} = \mathbf{0}$.	**5.** A typical vector **v** in Col A has the property $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector **v**, it is easy to determine if **v** is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector **v**, it takes time to determine if **v** is in Col A since you must row reduce $\begin{bmatrix} A & \mathbf{v} \end{bmatrix}$ first.
7. Nul $A = \{0\}$ if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbf{R}^m$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every **b**.

Review

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** are in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

EXAMPLE Determine whether each of the following is a vector space or provide a specific example to the contrary.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$ (b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$

(c) $S = \left\{ \begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

Solutions: (a) *Since* ____$= \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ is not in H, H is not a vector space.

(b) Rewrite $\begin{array}{l} x - y = 0 \\ y + z = 0 \end{array}$ as $x \begin{bmatrix} \quad \\ \quad \end{bmatrix} + y \begin{bmatrix} \quad \\ \quad \end{bmatrix} + z \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or as

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

(b) Since

$$\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S =$span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$ and is therefore a vector space by Theorem 1.

Kernal and Range of a Linear Transformation

Definition

A transformation T from a vector space V into a vector space W is a rule that assigns to each vector **x** in V a unique vector $T(\mathbf{x})$ in W, such that

- i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V and

  ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all **u** in tin V and all scalars c.

The ***kernal*** of T is the set of all vectors **u** in V such that $T(\mathbf{u}) = \mathbf{0}$. The ***range*** of T is the set of all vectors of the form $T(\mathbf{u})$ where **u** is in V.

EXAMPLE Let P_2 be the vector space of all polynomials of degree two or less and define the transformation $T : P_2 \to R^2$ such that $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}''(0) \end{bmatrix}$.

(a) Compute $T(\mathbf{p})$ if $\mathbf{p}(t) = 2t^2 + 3t + 1$.
(b) Verify that property (i) of a linear transformation holds here.

Solution: (a) $\mathbf{p}'(t) = 4t + 3$ and $\mathbf{p}''(t) = 4$. Then $T(\mathbf{p}) = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.

(b) Suppose **p** and **q** are polynomials of degree two or less. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})''(0) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}''(0)+\mathbf{q}''(0) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}''(0) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}''(0) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}).$$