MATH 114 - EXAM 3 - VERSION 1 - SOLUTIONS.

1. (a) $k=1$: $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

$k=2$: $\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

$k=3$: $\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$

(b) $S_1=a_1=\frac{1}{6}$

$S_2=a_1+a_2=\frac{1}{6}+\frac{1}{12}=\frac{1}{4}$

$S_3=a_1+a_2+a_3=\frac{1}{4}+\frac{1}{20}=\frac{3}{10}$

(c) $S_n=\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cancel{\cdots}+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)$

$=\frac{1}{2}-\frac{1}{n+2}$

(d) $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\frac{1}{2}$

2.

2 (a) $\sum_{k=2}^{\infty} \frac{k^3}{k^3+1}$ $\qquad \frac{k^2}{k^3+1} \sim \frac{1}{k}$, $\sum \frac{1}{k}$ diverges.

Compare with $\sum \frac{1}{k}$.

Limit comparison:

$$\lim_{k\to\infty} \frac{\frac{k^2}{k^3+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2}{k^3+1} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1$$

$\therefore \sum_{2}^{\infty} \frac{k^2}{k^3+1}$ diverges with $\sum \frac{1}{k}$. //

Direct comparison: $\frac{1}{k} \leq \frac{k^2}{k^3+1}$ not true.

$$\frac{1}{2k} \leq \frac{k^2}{k^3+1} \iff k^3+1 \leq 2k^3 \iff k^3 \geq 1$$

$\therefore$ True if $k \geq 1$.

For all $k$, $\frac{1}{2k} \leq \frac{k^2}{k^3+1}$. Since $\sum \frac{1}{2k}$ $= \frac{1}{2} \sum \frac{1}{k}$ diverges, so does $\sum \frac{k^2}{k^3+1}$



V1

2(b) $\sum_{k=2}^{\infty} \frac{1}{1+3^k}$ $\qquad \frac{1}{1+3^k} \sim \frac{1}{3^k} = \left(\frac{1}{3}\right)^k$

compare with $\sum \left(\frac{1}{3}\right)^k$
which converges.

Limit comp: $\lim_{k \to \infty} \frac{\frac{1}{1+3^k}}{\frac{1}{3^k}} = \lim_{k \to \infty} \frac{3^k}{1+3^k} = 1$

$\therefore$ Series converges with $\sum \left(\frac{1}{3}\right)^k$.

Direct Comparison:

$\frac{1}{1+3^k} \leq \frac{1}{3^k} \iff 3^k \leq 1+3^k \iff 0 \leq 1.$

$\therefore$ for all $k$, $\frac{1}{1+3^k} \leq \frac{1}{3^k}$ so

since $\sum \left(\frac{1}{3}\right)^k$ converges so does $\sum \frac{1}{1+3^k}$.



V1

3 (a) $\sum_{k=2}^{\infty} \frac{10^{5k}}{k!}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{10^{5(n+1)}}{(n+1)!} \cdot \frac{n!}{10^{5n}}$

$= \lim_{n\to\infty} \frac{10^5}{n+1} = 0 < 1$ $\therefore$ Series converges.

(b) $\sum_{k=2}^{\infty} \left(\frac{k}{2k+1}\right)^{2k}$

Root Test: $|a_n|^{1/n} = \left|\left(\frac{n}{2n+1}\right)^{2n}\right|^{1/n} = \left(\frac{n}{2n+1}\right)^2 \to \frac{1}{4}$

as $n \to \infty$. Since $\frac{1}{4} < 1$ Series converges.



V1

4. (a) $\sum_{k=0}^{\infty} 8^{-k} 3^{k+1} = \sum_{k=0}^{\infty} 3 \cdot \left(\frac{3}{8}\right)^k = \frac{3}{1-\frac{3}{8}} = \frac{24}{5}$

Series converges to $\frac{24}{5}$.

(b) $\sum_{k=1}^{\infty} \frac{k}{k+\ln k}$ $\quad \lim_{k\to\infty} \frac{k}{k+\ln k} = 1 \neq 0$

Series diverges by Divergence Test.

5. (a) $\sum_{k=1}^{\infty} \left| \frac{\cos(k^2)}{k^3} \right| \le \sum_{k=1}^{\infty} \frac{|\cos(k^2)|}{k^3} \le \sum_{k=1}^{\infty} \frac{1}{k^3}$

which converges by p-series test $(p=3>1)$

Hence the series converges absolutely.

(b) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k+\sqrt{k}}$

(1) $\frac{1}{k+\sqrt{k}} \ge 0$ all $k$

(2) $\frac{1}{k+\sqrt{k}}$ is decreasing

(3) $\frac{1}{k+\sqrt{k}} \to 0$ as $k \to \infty$.

By A.S.T, series converges.

$\sum_{k=1}^{\infty} \left| (-1)^k \frac{1}{k+\sqrt{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{k+\sqrt{k}}$

L.C. to $\sum \frac{1}{k}$

$\lim \frac{\frac{1}{k+\sqrt{k}}}{\frac{1}{k}} = \lim \frac{k}{k+\sqrt{k}} = 1$

Series converges conditionally.

VI

# MATH 114 – EXAM 3 – VERSION 2 – SOLUTIONS.

1. (a) $a_1 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}$

$a_2 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}}$

$a_3 = \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{9}}$ //

(b) $S_1 = a_1 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}$ //

$S_2 = a_1 + a_2 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}}$

$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{7}}$ //

$S_3 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{9}}$

$= \frac{1}{\sqrt{3}} - \frac{1}{3}$ //

(c) $S_n = \sum_{k=1}^{n} \left( \frac{1}{\sqrt{2k+1}} - \frac{1}{\sqrt{2k+3}} \right)$

$= \left(\frac{1}{\sqrt{3}} - \cancel{\frac{1}{\sqrt{5}}}\right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \cancel{\frac{1}{\sqrt{7}}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{\sqrt{2n+1}}} - \frac{1}{\sqrt{2n+3}}\right)$

$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2n+3}}$ //

(d) $\sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{2k+1}} - \frac{1}{\sqrt{2k+3}} \right) = \lim_{n \to \infty} \left( \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2n+3}} \right) = \frac{1}{\sqrt{3}}$ //



V2

2. (a) $\sum_{k=2}^{\infty} \sqrt{\frac{k^2}{k^4+1}}$ $\quad \sqrt{\frac{k^2}{k^4+1}} \sim \sqrt{\frac{k^2}{k^4}} \sim \frac{1}{k}$

compare with $\sum \frac{1}{k}$ which diverges

Limit comp Test:

$$\lim_{n\to\infty} \frac{\sqrt{\frac{k^2}{k^4+1}}}{\frac{1}{k}} = \lim_{n\to\infty} \sqrt{\frac{k^2}{k^4+1}} \cdot \sqrt{\frac{k^2}{1}} = \lim_{k\to\infty} \sqrt{\frac{k^4}{k^4+1}} = 1$$

Series diverges with $\sum \frac{1}{k}$.

Direct Comparison: Would like $\frac{1}{k} \le \sqrt{\frac{k^2}{k^4+1}}$

but this is not true. So try

$$\frac{1}{2k} \le \sqrt{\frac{k^2}{k^4+1}} \iff \frac{1}{4k^2} \le \frac{k^2}{k^4+1} \iff$$

$k^4+1 \le 4k^4 \iff 1 \le 3k^4$ True for all $k$.

Hence for all $k$ $\frac{1}{2k} \le \sqrt{\frac{k^2}{k^4+1}}$ and since

$\sum \frac{1}{2k} = \frac{1}{2} \sum \frac{1}{k}$ diverges, so does $\sum \sqrt{\frac{k^2}{k^4+1}}$



V2

26) $\sum_{k=2}^{\infty} \frac{3^k}{10+5^k}$ $\frac{3^k}{10+5^k} \sim \frac{3^k}{5^k} = \left(\frac{3}{5}\right)^k$

Compare with $\sum \left(\frac{3}{5}\right)^k$ which converges.

Limit comparison:

$$\lim_{n\to\infty} \frac{\frac{3^n}{10+5^n}}{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{3^n}{10+5^n} \cdot \frac{5^n}{3^n} = \lim_{n\to\infty} \frac{5^n}{5^n+10} = 1$$

Series converges with $\sum \left(\frac{3}{5}\right)^n$.

Direct comparison: $\frac{3^k}{10+5^k} \le \frac{3^k}{5^k} \Longleftrightarrow$

$3^k \cdot 5^k \le 3^k (10+5^k) \Longleftrightarrow 5^k \le 10+5^k \Longleftrightarrow$

$0 \le 10$ so true for all $k$.

Since $\sum \left(\frac{3}{5}\right)^k$ converges and $\frac{3^k}{10+5^k} \le \left(\frac{3}{5}\right)^k$

$\sum \frac{3^k}{10+5^k}$ converges.

V2

3 (a) Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(2(n+1))!}\cdot\frac{(2n)!}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty}\frac{n+1}{(2n+2)(2n+1)}$$

$$= 0 < 1.$$

Series converges by Ratio Test

(b) Root test

$$\sum_{k=2}^{\infty}\left(\frac{2k}{k+1}\right)^{k/2} \quad \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left[\left(\frac{2n}{n+1}\right)^{n/2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{2n}{n+1}\right)^{1/2} = 2^{1/2} > 1$$

Series diverges by Root Test



V2

4.(a) $\sum_{k=2}^{\infty} \frac{7}{4^k} = \frac{7}{4^2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{7}{16} \cdot \frac{1}{1-\frac{1}{4}}$

$= \frac{7}{16} \cdot \frac{4}{3} = \frac{7}{12}$ // Since its sum is finite, the series converges.

(b) $\sum_{k=1}^{\infty} \frac{k}{k+\sqrt{k}}$ $\quad \lim_{k\to\infty} \frac{k}{k+\sqrt{k}} = 1 \neq 0$

series diverges by Divergence Test.

5 (a) $\sum_{k=1}^{\infty} \left|\frac{\sin(k^3)}{k^3}\right| = \sum_{k=1}^{\infty} \frac{|\sin(k^3)|}{k^3} \le \sum_{k=1}^{\infty} \frac{1}{k^3}$

Since $\sum \frac{1}{k^3}$ converges ($p$-series, $p=3>1$)

By Direct Comparison Test, $\sum \frac{\sin(k^3)}{k^3}$ converges absolutely.

(b) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k+\ln k}$

(1) $\frac{1}{k+\ln k} \ge 0$ all $k$

(2) $\frac{1}{k+\ln k}$ decreases

(3) $\frac{1}{k+\ln k} \to 0$ as $k \to \infty$.

∴ By AST, series converges.

$$\sum_{k=1}^{\infty}\left|(-1)^k \frac{1}{k+\ln k}\right| = \sum_{k=1}^{\infty} \frac{1}{k+\ln k}.$$

Limit comparison to $\sum \frac{1}{k}$ which diverges.

$$\lim_{k\to\infty} \frac{\left(\frac{1}{k+\ln k}\right)}{\left(\frac{1}{k}\right)} = \lim_{k\to\infty} \frac{k}{k+\ln k} = 1$$

Series diverges with $\sum \frac{1}{k}$.

$\therefore \sum_{k=1}^{\infty} (-1)^k \frac{1}{k+\ln k}$ converges conditionally //

. ons the section



V2