QUIZ 13 – 9.1, 9.2

Exam 3 – Monday 4-29 (info on web)

Oral Reviews – R and F this week.

Power Series: Given $f(x)$ can we write $f(x)=\sum_{k=0}^{\infty} c_k (x-a)^k$?

① If so $c_k = \frac{1}{k!} f^{(k)}(a)$

② Any power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ has a <u>radius of convergence</u> R such that series converges absolutely if $|x-a|<R$, or on $(a-R, a+R)$.

③ We know some $f(x)$ ~~ha~~ that have a power series expansion.

<u>e.g.</u> $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x|<1$.

<u>e.g.</u> Suppose $f(x) = \frac{x}{(1-x^2)^2}$. Write f as a power series.

$$\frac{x}{(1-x^2)^2} = x \cdot \frac{1}{(1-x^2)^2}$$

What if it were $\frac{1}{(1-x)^2}$? $\frac{d}{dx}\left(\frac{1}{1-x}\right)$

$$= \frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

<u>Idea:</u> $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{0}^{\infty} x^k\right)$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}(x^k) = \sum_{k=0}^{\infty} k x^{k-1} = \sum_{k=1}^{\infty} k x^{k-1}$$

$\therefore \; \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$.

$$\therefore \; \frac{1}{(1-x^2)^2} = \sum_{k=1}^{\infty} k (x^2)^{k-1} = \sum_{k=1}^{\infty} k x^{2k-2}$$

$$\therefore \; \frac{x}{(1-x^2)^2} = \sum_{k=1}^{\infty} k(x)(x^{2k-2}) = \sum_{k=1}^{\infty} k x^{2k-1}$$

for $|x|<1$ (this is a seperate calculation)

9.3 Taylor Series.

What about e^x?, $\sin(x)$?, $\tan(x)$?

Def: Given $f(x)$ and a center a, the Taylor series of $f(x)$ about (or at) a is $\sum_{k=0}^{\infty} c_k (x-a)^k$ where $c_k = \frac{1}{k!} f^{(k)}(a)$

(Note that we must be able to differentiate f as many times as we want.)

e.g. $f(x) = e^x$, $a=0$

$f'(x) = e^x$ $\quad f^{(k)}(x) = e^x$

$f''(x) = e^x$ $\quad \therefore f^{(k)}(0) = e^0 = 1$ for all k.

$f'''(x) = e^x$

$\vdots$ $\quad \therefore c_k = \frac{1}{k!} f^{(k)}(0) = \frac{1}{k!}$

Taylor series for e^x at $a=0$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k \qquad \left(\sum_{k=0}^{\infty} \frac{1}{k!} (x-0)^k \right)$$

Q: What is rad. of conv for this series?

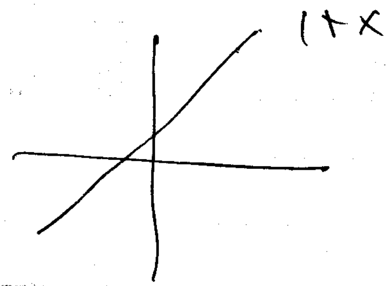
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = |x|\left(\frac{1}{n+1}\right) \to 0 < 1$$

for all x.

So series converges for all $x \in (-\infty, \infty)$ so $R = \infty$.

Q: What do partial sums look like?

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

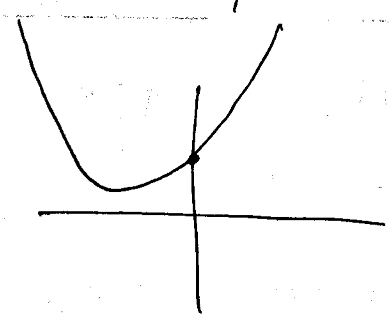


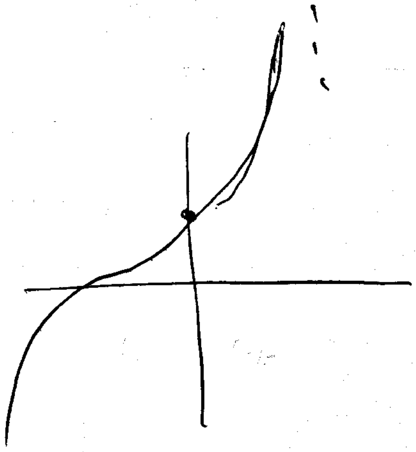
$P_1(x) = 1 + x$

$P_2(x) = 1 + x + \frac{1}{2}x^2$

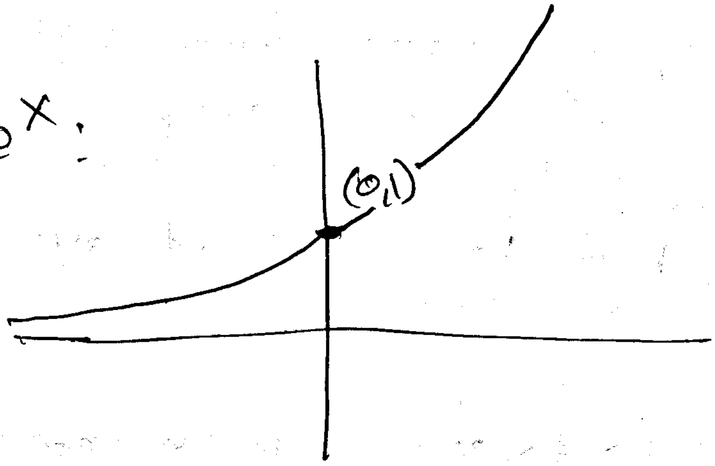
$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$\vdots$





e^x:



<u>Q</u>: Is it true that

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \ ?$$

<u>e.g.</u> $f(x) = \sin(x)$ at $\underline{a=0}$. $f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$.

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

1

0

-1

0

$\vdots$

$$C_k = \frac{1}{k!} f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{1}{k!}(-1) & \text{if } \frac{k-1}{2} \text{ odd} \\ \frac{1}{k!} & \text{if } \frac{k-1}{2} \text{ even.} \end{cases}$$

So $\sin(x) \sim 1\cdot x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{7!}x^7 + \cdots$

To write this: $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

radius of convergence $= \infty$.

partial sums are polynomials with odd degree

<u>e.g.</u> binomial series. $f(x) = (1+x)^p$, $a=0$.

$f(x) = (1+x)^p$ $\qquad f(0) = 1$

$f'(x) = p(1+x)^{p-1}$ $\qquad f'(0) = p$

$f''(x) = p(p-1)(1+x)^{p-2}$ $\qquad f''(0) = p(p-1)$

$f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$ $\qquad f'''(0) = p(p-1)(p-2)$

$\vdots$

$f^{(k)}(x) = p(p-1)(p-2)\cdots(p-(k-1))(1+x)^{p-k}$

$\vdots$

$f^{(k)}(0) = p(p-1)\cdots(p-k+1)$

$$c_k = \frac{1}{k!} f^{(k)}(0) = \frac{p(p-1)\cdots(p-k+1)}{k!} = \binom{p}{k}$$

$\frac{p!}{k!(p-k)!}$

e.g. #12) $\ln(1+x)$; $a=0$.

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$ $\qquad \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$f(0) = 0 \qquad c_0 = 0$

$f'(0) = 1 \qquad c_1 = 1$

$f''(0) = -1 \qquad c_2 = \frac{1}{2!}f^{(2)}(0) = -\frac{1}{2}$

$f'''(0) = 2 \qquad c_3 = \frac{1}{3!}f^{(3)}(0) = \frac{1}{6}\cdot 2 = \frac{1}{3}$

$$\sum_{k=0}^{\infty} c_k x^k = 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$\ln(1+x) \sim \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}$$

#20) $f(x) = \frac{1}{x}$ $\quad a=2$.

$f(x) = \frac{1}{x} = x^{-1} \qquad f(2) = \frac{1}{2} \qquad c_0 = \frac{1}{2}$

$f'(x) = -x^{-2} \qquad f'(2) = -\frac{1}{4} \qquad c_1 = -\frac{1}{4}$

$f''(x) = 2x^{-3} \qquad f''(2) = \frac{1}{4} \qquad c_2 = \frac{1}{2!}\cdot\frac{1}{4} = \frac{1}{8}$

$f'''(x) = -6x^{-4} \qquad f'''(2) = -\frac{3}{8} \qquad c_3 = \frac{1}{3!}\cdot -\frac{3}{8} = -\frac{1}{16}$

$$\sum_{k=0}^{\infty} c_k(x-2)^k = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \cdots$$

#30) $f(x) = (1+x)^{1/2}$ approximate $(1.06)^{1/2}$

$$(1+x)^{1/2} \sim \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Binomial series

$$(1+x)^{p} = \sum_{k=0}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} x^k$$

$$\boxed{0! = 1}$$

$$c_k = \frac{1}{k!} f^{(k)}(a)$$

$k=0$: $c_0 = 1$

$k=1$: $c_1 = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$

$k=2$: $c_2 = \frac{1}{2!} \cdot \left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{-1}{2} = \frac{-1}{8}$

$k=3$: $c_3 = \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) = \frac{1}{2\cancel{6}}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{\cancel{3}}{2}\right) = \frac{1}{16}$

$$(1+x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Idea: $x = 0$ series is exact

x near 0 series is a good approx.

$$(1.06)^{1/2} \approx 1 + \frac{1}{2}(.06) - \frac{1}{8}(.06)^2 + \frac{1}{16}(.06)^3$$

$x = .06$ $\qquad = 1.0295635$