Quiz 13 9.1, 9.2
Exam 3 – Mon 4-29 8.1–8.6
No calculators, 3x5 formula card
Oral Reviews, Thurs / Fri. Check web.
Final Exam – May 8 1^{30}–4^{15}.

9.2 Properties of Power Series.

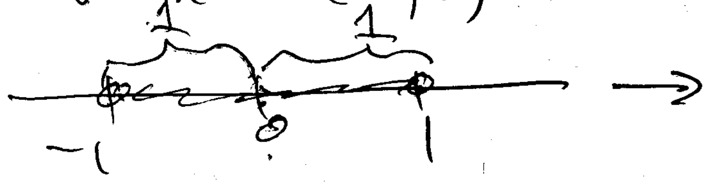
Can we write any $f(x)$ as $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$
where a (the center) is given?
We have seen that $c_k = \frac{1}{k!} \underbrace{f^{(k)}(a)}_{k^{th} \text{ derivative of } f \text{ at } a}$.

A series of the form $\sum_{k=0}^{\infty} c_k (x-a)^k$ is a
power series.

Q: Given $\sum_{k=0}^{\infty} c_k (x-a)^k$ for which x does it
converge?

e.g. $\sum_{k=0}^{\infty} x^k$ converges if $|x|<1$ and diverges
if $|x| \geq 1$.

$a = 0$ $(-1,1)$
$c_k = 1$ Int of conv $= (-1,1)$



<u>Idea</u> $\sum_{k=0}^{\infty} c_k (x-a)^k$ looks like geometric series.

The coeffs will then help series converge or help it diverge.

<u>e.g.</u> $$\sum_{k=0}^{\infty} \left(\frac{x+5}{2}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k (x+5)^k$$

so $a = -5$ $\quad c_k = \left(\frac{1}{2}\right)^k$

Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{1}{2}\right)^{n+1} (x+5)^{n+1}}{\left(\frac{1}{2}\right)^n (x+5)^n}\right|$$

$$= \frac{1}{2}|x+5| \; ; \; \boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}|x+5| < 1}$$

So $\sum_{k=0}^{\infty} \left(\frac{x+5}{2}\right)^k$ converges for $\frac{1}{2}|x+5| < 1$

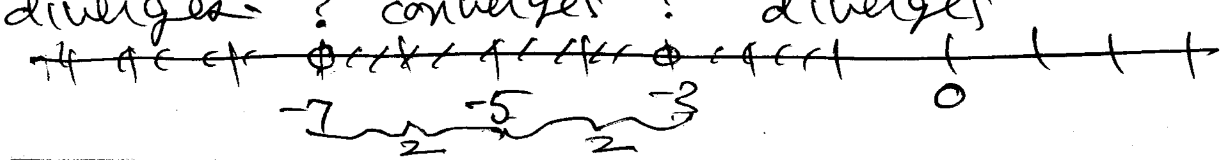
or $|x+5| < 2 \longrightarrow -2 < x+5 < 2$

$\qquad\qquad\qquad\qquad -7 < x < -3$

Converges for $x \in (-7, -3)$.

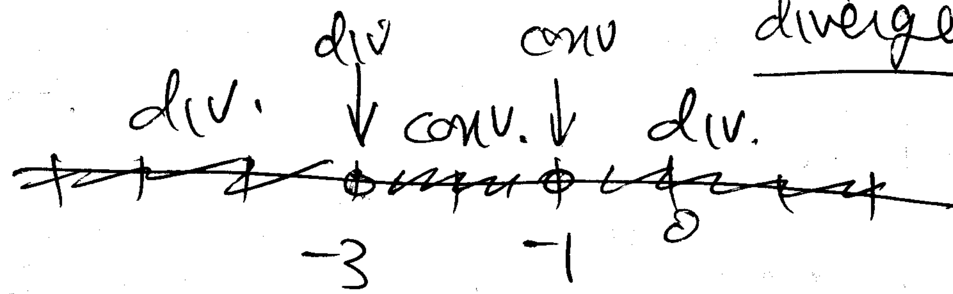
Diverges for ~~x~~ $|x+5| > 2$, i.e.

$x \in (-\infty, -7) \cup (-3, \infty)$

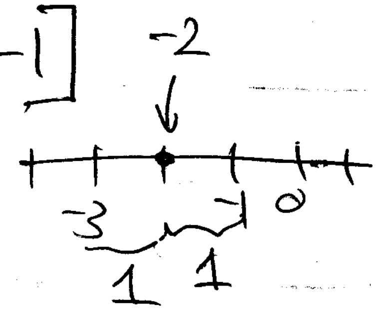


$\underline{x=-3}$: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k(-3+2)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ $\underline{\text{diverges}}$

~~Series co~~



Interval of convergence $= (-3, -1]$



$\underline{\text{e.g.}}$ $\displaystyle\sum_{k=0}^{\infty} \frac{3^k x^k}{k!}$ $\quad a = 0$

$C_k = \dfrac{3^k}{k!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{3^n}\, 3 \cdot \cancel{x^n}\, x}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{3^n}\,\cancel{x^n}}\right|$$

$$= \left|\frac{3x}{n+1}\right| = \frac{3}{n+1}|x| \longrightarrow 0 < 1$$

So $\displaystyle\sum_{0}^{\infty} \frac{3^k x^k}{k!}$ converges for all x.

Interval of convergence $= (-\infty, \infty)$

$x=-7 \longrightarrow \sum_{k=0}^{\infty}\left(\frac{x+5}{2}\right)^k = \sum_{k=0}^{\infty}\left(\frac{-2}{2}\right)^k = \sum_{k=0}^{\infty}(-1)^k$ diverges.

$x=-3 \longrightarrow \sum_{k=0}^{\infty}\left(\frac{x+5}{2}\right)^k = \sum_{k=0}^{\infty}\left(\frac{2}{2}\right)^k = \sum_{k=0}^{\infty} 1$ diverges.

$\therefore$ $\boxed{\text{Interval of convergence} = (-7,-3).}$

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k}$ $\quad a=-2$, $c_k = \frac{(-1)^k}{k}$

Ratio: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x+2)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n (x+2)^n}\right|$

$= \frac{n}{n+1}|x+2| \longrightarrow |x+2|$ as $n\to\infty$. $\quad -1<x+2<1$, $-3<x<-1$

Series converges if $|x+2|<1 \rightarrow x\in(-3,-1)$

Series diverges if $|x+2|>1 \rightarrow x\in(-\infty,-3)\cup(-1,\infty)$.

$|x+2|=1 \longrightarrow x=-1$, $x=-3$

$x=-1$: $\sum_{k=1}^{\infty}\frac{(-1)^k(-1+2)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ converges by A.S.T.

e.g. $\sum_{k=0}^{\infty} k!\,(x-4)^k$ $\quad a=4$

$c_k = k!$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n}\right| = (n+1)\,|x-4|$$

$\longrightarrow \infty$ for all x except $x=4$

So $\sum_{0}^{\infty} k!\,(x-4)^k$ converges only at $\underline{x=4}$.

interval of convergence $= [4,4]$

Given $\sum_{k=0}^{\infty} c_k (x-a)^k$ there are 3 possibilities

① There is an $R>0$ called the <u>radius of convergence</u> such that series converges absolutely when $|x-a|<R$ (or $x\in(a-R,a+R)$) and diverges when $|x-a|>R$.

② Series converges absolutely for all x. We say $R=\infty$ in this case.

③ Series converges only when $x=a$. We say $R=0$ in this case.

The interval of convergence may or may not include one or both of endpoints, i.e. interval of convergence could be:

$$\left.\begin{array}{l}(a-R, a+R)\\ [a-R, a+R)\\ (a-R, a+R]\\ [a-R, a+R].\end{array}\right\}$$ depends on convergence of series at $x=a+R$ and $x=a-R$.

#10 $\sum (-1)^k \frac{x^k}{5^k}$

#12 $\sum (-1)^k \frac{k(x-4)^k}{2^k}$

#14 $\sum \frac{k^k x^k}{(k+1)!}$

Find rad of convergence + interval of convergence.

$$\left|\frac{\cancel{(-1)^{n+1}} x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{\cancel{(-1)^n} x^n}\right| = \left|\frac{x}{5}\right| = \frac{1}{5}|x| < 1$$

or $|x|<5$ $R=5$ $|x-a|<R$

$(-5,5)$ $x=-5 \to \sum (-1)^k \frac{(-5)^k}{(5)^k} = \sum (-1)^k(-1)^k = \sum 1$ diverges.

int of conv. $x=5 \to \sum (-1)^k$ diverges

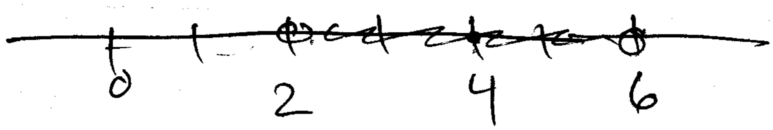
12) $\left|\dfrac{a_{n+1}}{a_n}\right| = \left| \dfrac{(-1)^{n+1}\dfrac{(n+1)(x-4)^{n+1}}{2^{n+1}}}{(-1)^n \dfrac{n(x-4)^n}{2^n}} \right|$

$$= \left| \frac{(n+1)(x-4)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x-4)^n} \right|$$

$$= \left|\cancel{\frac{n+1}{n}}\right| \left(\frac{1}{2}\right)|x-4| \xrightarrow[n\to\infty]{} \frac{1}{2}|x-4|$$

$$\frac{1}{2}|x-4| < 1 \longrightarrow |x-4| < 2$$

Radius of conv. $= 2$



$x=2$: $\sum (-1)^k \dfrac{k(2-4)^k}{2^k} = \sum (-1)^k \cdot k \cdot \dfrac{(-2)^k}{2^k}$

$= \sum k \left(\dfrac{(-1)(-2)}{2}\right)^k = \sum k$ diverges

$x=6$: $\sum (-1)^k \dfrac{k(6-4)^k}{2^k} = \sum k(-1)^k$ diverges.

Int of conv $= (2, 6)$.

Sometimes we can evaluate power series.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x|<1$$

e.g. Find power series for $\frac{x^3}{1-x} = (x^3)\frac{1}{1-x}$

$$= x^3 \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+3} = \sum_{k=3}^{\infty} x^k$$

$$\frac{x^3}{1-x} = \sum_{k=3}^{\infty} x^k \quad \text{if } |x|<1.$$

e.g. Power series for $\frac{x^3}{1-x^2} = x^3 \cdot \frac{1}{1-x^2}$

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}$$

$a=0$

$c_k = 1,0,1,0,1,0,1,\ldots$

$$= 1 + x^2 + x^4 + x^6 + \cdots$$

if $|x^2|<1$

$|x|^2<1$

$|x|<1$

$$= 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3 + 1\cdot x^4 + \cdots$$

$$\frac{x^3}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} \cdot x^3 = \sum_{k=0}^{\infty} x^{2k+3}$$

for $|x|<1$

e.g. $\ln(1-x) \leftarrow$ Power series for.

$$\cancel{\frac{d}{dx}}\int \frac{1}{1-x}\,dx = \int \sum_{k=0}^{\infty} x^k\,dx \qquad (|x|<1)$$

$$-\ln|1-x| = \sum_{k=0}^{\infty} \int x^k\,dx = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} + c$$

(integrating term-by-term)

$x=0$: $\quad 0 \qquad\qquad 0 + c$

so $c=0$

$$\ln|1-x| = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \text{for } |x|<1$$

Let $x=-1$: $\ln(2) = \sum_{k=0}^{\infty} (-1)\frac{(-1)^{k+1}}{k+1}$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

e.g. $\frac{d}{dx}\left(\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k\right)$

$$\frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}\left(\frac{d}{dx}x^k\right)$$

$$= \sum_{k=0}^{\infty} k x^{k-1} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} \quad \text{if } |x| < 1$$

$x = \frac{1}{2}$ $\quad \frac{1}{(1-\frac{1}{2})^2} = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = 4$

$$1 + 1 + \frac{3}{4} + \frac{1}{2} + \cdots = 4$$