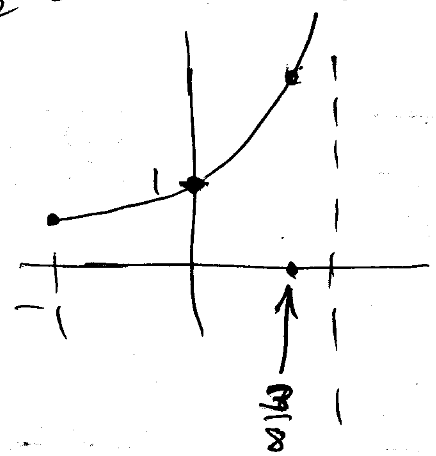
Quiz 12 ~~8.4~~, 8.5, 8.6.

Power series.

e.g. $\sum_{k=0}^{\infty} x^k = f(x).$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x|<1, \text{ that is, if } x \in (-1,1).$$



(1) We have taken an ordinary function $f(x) = \frac{1}{1-x}$ and written it as an infinite series.

(2) What is the point? The partial sums of $\sum_{k=0}^{\infty} x^k$ are polynomials.

$$f\left(\tfrac{3}{8}\right) = \frac{1}{1-\frac{3}{8}} = \frac{8}{5} = \underline{1.6}$$

$S_0 = 1$

$S_1 = 1+x$

$S_2 = 1+x+x^2$

$S_3 = 1+x+x^2+x^3$

$$\boxed{\text{So we can take } f(x) = \frac{1}{1-x} \text{ and approximate it by polynomials}}$$

$S_0\left(\frac{3}{8}\right) = 1$

$S_1\left(\frac{3}{8}\right) = 1 + \frac{3}{8} = \frac{11}{8} = 1.375$

$S_2\left(\frac{3}{8}\right) = 1 + \frac{3}{8} + \frac{9}{64} = \frac{11}{8} + \frac{9}{64} = \frac{97}{64} = 1.515625$

$S_3\left(\frac{3}{8}\right) = \frac{97}{64} + \frac{27}{512} = 1.568\ldots$

(3) Can I do this with any function?

eg $f(x) = \ln(x)$? $f(x) = \cos(x)$?

$f(x) = e^x$?

If so what would the polynomials look like?

9.1 Taylor Polynomials.

In general we ask: Can I write

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k \ ?$$

a = center

c_k = sequence of coefficients.

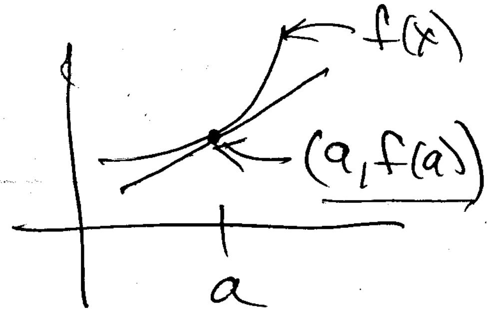
So

$$P_n(x) = \sum_{k=0}^{n} c_k (x-a)^k$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$

Given $f(x)$ what must the c_k be?

Idea:



slope of tan line at a $= f'(a)$

eqn of tangent line:

point $y - f(a) = f'(a)(x-a)$

$$y = f(a) + f'(a)(x-a)$$

This is best straight-line approx for $f(x)$ near a.

What about approximating $f(x)$ with a quadratic polynomial?

$$P_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2$$

Want: exact approx at a.

$$P_2(a) = c_0 = f(a)$$

$$P_2'(a) = c_1 = f'(a)$$

$$P_2'(x) = c_1 + 2c_2(x-a)$$

$$P_2''(a) = 2c_2 = f''(a) \longrightarrow c_2 = \frac{1}{2}f''(a).$$

$$P_2''(x) = 2c_2$$

Conclusion: $P_2(x) = f(a) + f'(a)(x-a) + \underline{\tfrac{1}{2}f''(a)(x-a)^2}$

is the best quadratic approx to f near a.

What about a cubic?

$$P_3(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3$$

$$P_3'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2$$

$$P_3''(x) = 2c_2 + 2\cdot 3\, c_3(x-a)$$

$$P_3'''(x) = 2\cdot 3\, c_3 = f'''(a) \longrightarrow c_3 = \frac{1}{3\cdot 2} f'''(a).$$

What about for degree n?

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

$$\boxed{c_k = \frac{1}{k!} f^{(k)}(a)}$$

$$c_4 = \frac{1}{4\cdot 3\cdot 2} f^{(4)}(a)$$

$$c_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2} f^{(5)}(a)$$

$$\vdots$$

Main point: $P_n(x) \approx f(x)$ if <u>x is near a</u>.

<u>eg</u> #8) $f(x) = x^{1/2}$; <u>$a=4$</u>; approx $\sqrt{3.9}$

$$P_1(x) = f(a) + f'(a)(x-a) \qquad f(4) = 4^{1/2} = 2$$

$$= 2 + \tfrac{1}{4}(x-4) \qquad f'(x) = \tfrac{1}{2}x^{-1/2}$$

$$P_1(3.9) = 2 + \tfrac{1}{4}\underbrace{(3.9-4)}_{-.1} \qquad f'(4) = \tfrac{1}{2}4^{-1/2} = \tfrac{1}{4}$$

$$= 2 - \tfrac{1}{40} \qquad f''(x) = -\tfrac{1}{4}x^{-3/2}$$

$$= \tfrac{79}{40} = 1.975 \approx f(3.9) \qquad f''(4) = -\tfrac{1}{4}4^{-3/2} = \tfrac{-1}{32}.$$

$$\cancel{P_2(3.9)=}\ P_2(x) = 2 + \tfrac{1}{4}(x-4) + \left(\tfrac{1}{2}\right)\left(\tfrac{-1}{32}\right)(x-4)^2$$

$$= 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2$$

$$P_2(3.9) = 2 + \tfrac{1}{4}\left(-\tfrac{1}{10}\right) - \tfrac{1}{64}\left(\tfrac{1}{100}\right) = \tfrac{79}{40} - \tfrac{1}{6400} \cong 1.9748\ldots$$

$$\sqrt{3.9} = 1.9748417 66\ldots$$

e.g. $f(x) = \ln(x)$; $a = 1$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3}$$

$$f^{(4)}(x) = -6x^{-4}$$

$$\vdots$$

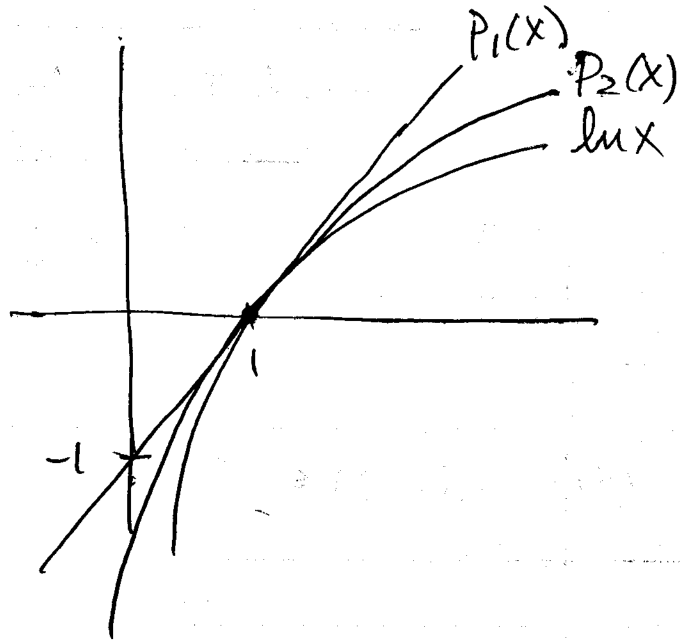
$$P_1(x) = f(1) + f'(1)(x-1)$$

$$= 0 + (1)(x-1) = x-1$$

$$P_2(x) = (x-1) + \tfrac{1}{2}f''(1)(x-1)^2$$

$$= x-1 + \left(-\tfrac{1}{2}(x-1)^2\right)$$

$$= x-1 - \tfrac{1}{2}(x-1)^2$$



$$P_3(x) = \left[x-1-\tfrac{1}{2}(x-1)^2\right] + \left(\tfrac{1}{6}\right)f'''(1)(x-1)^3$$

$$= \left[x-1-\tfrac{1}{2}(x-1)^2\right] + \tfrac{1}{3}(x-1)^3$$