Quiz 11 – 8.5/8.6

Exam 3 – April 29 (two weeks) Ch 8 (plus more)

Final Exam – May 8

$\sum_{k=1}^{\infty} a_k$

① Two sequences: terms a_k
partial sums $S_n = \sum_{k=1}^{n} a_k$.

② Usually we can't sum a series
except: ⓐ geometric series $\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$ if $|r|<1$.
ⓑ telescoping series

$$\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k \neq \cancel{\frac{1}{1-\frac{2}{3}}}$$

$$= \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k - \left(\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1\right)$$

$$= \frac{1}{1-\frac{2}{3}} - 1 - \frac{2}{3} = 3 - 1 - \frac{2}{3} = 2 - \frac{2}{3} = \frac{4}{3} //$$

$$\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k = \left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \cdots\right)$$

$$= \left(\frac{2}{3}\right)^2 \underbrace{\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right)}_{\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k}$$

$$= \left(\frac{2}{3}\right)^2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$= \frac{4}{9} \cdot \frac{1}{1-\frac{2}{3}} = \frac{4}{9} \cdot 3 = \frac{4}{3} //$$

③ We settle for deciding whether series converges or not.

ⓐ Divergence test: If $\lim_{k \to \infty} a_k \neq 0$, $\sum a_k$ diverges.

ⓑ Integral test.

ⓒ Comparison tests. (direct and limit comparison)

ⓓ Ratio Test $\rightarrow \frac{a_{k+1}}{a_k} \to r$

ⓔ Root Test. $\rightarrow (a_k)^{1/k} = \sqrt[k]{a_k} \to r.$

ⓕ p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ $\begin{cases} p \leq 1 & \text{diverges} \\ p > 1 & \text{converges} \end{cases}$

$p > 0$

8.6 Alternating series

Idea: For all of our tests on $\sum a_k$ we assume $a_k \geq 0$. This is because S_n is increasing.

What if we allow pos and neg terms? All bets are off.

e.g. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

S_n: $S_1 = 1$ $\qquad S_6 = .62$

$S_2 = .5$ $\qquad S_7 = .76$

$S_3 \cong .83$ $\qquad S_8 = .63$

$S_4 \cong .58$ $\qquad \vdots$

$S_5 \cong .78$

An *alternating series* has the form

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{where } a_k \geq 0.$$

$$= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

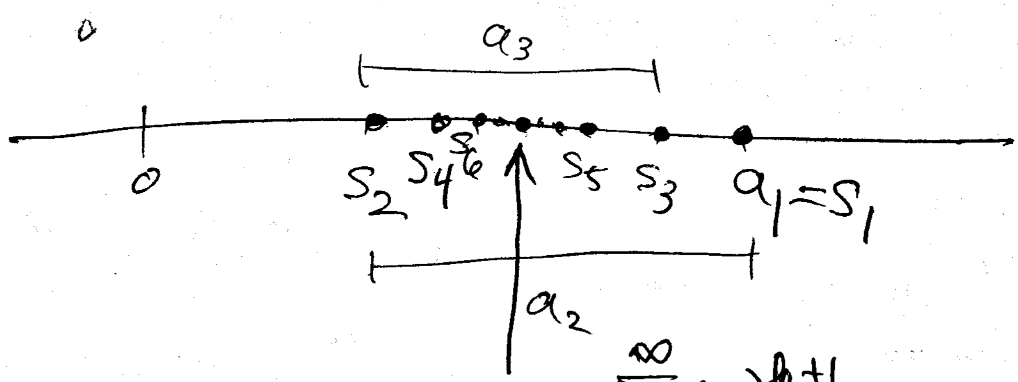
Thm: $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k \geq 0$ converges if

(a) a_k is decreasing, i.e. $a_k \geq a_{k+1} \geq a_{k+2} \geq \cdots$

(b) $\lim_{k \to \infty} a_k = 0$.

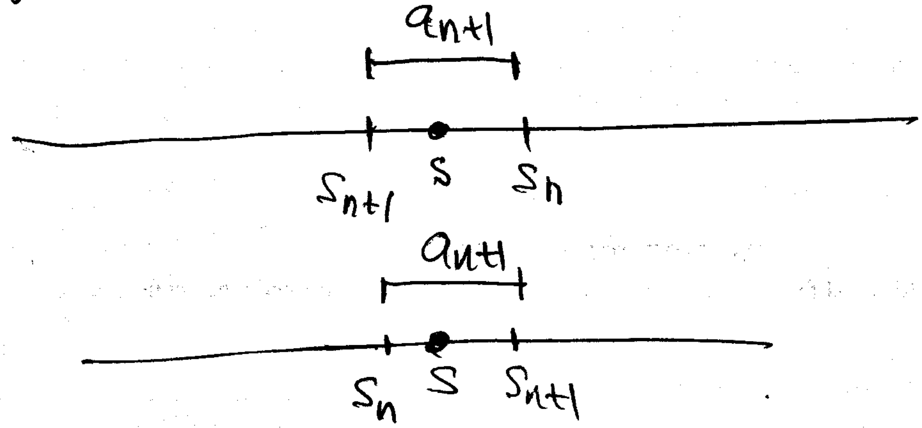
Note: It does not matter in this case how fast $a_k \to 0$! This is because signs alternate

Why? Look at $S_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$



$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

We get a bonus!



So $|S - S_n| \le a_{n+1}$

$$\boxed{\text{Formula: } |S - S_n| \le a_n}$$

e.g. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$ $a_k = \dfrac{1}{k^2+10}$

converges by A.S.T. (Alternat. Series Test)

$a_k \ge 0$, a_k decreasing (since k^2+10 is incr.)

$\displaystyle\lim_{k\to\infty} \frac{1}{k^2+10} = 0$

e.g. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{1/3}}$ $a_k = \dfrac{1}{k^{1/3}}$

converges by Alternating Series Test (A.S.T.)

$a_k \ge 0$, a_k decreasing ($k^{1/3}$ increasing)

$\displaystyle\lim_{k\to\infty} \frac{1}{k^{1/3}} = 0$

eg $\sum_{k=2}^{\infty}(-1)^k \frac{1}{\ln(k)}$ $\frac{1}{\ln k} \geq 0$

converges by Alt. Series Test. $\frac{1}{\ln k}$ is decreasing

$\lim_{k\to\infty} \frac{1}{\ln k} = 0$

eg $\sum_{k=2}^{\infty}(-1)^k \ln(k)$ diverges by Divergence Test.

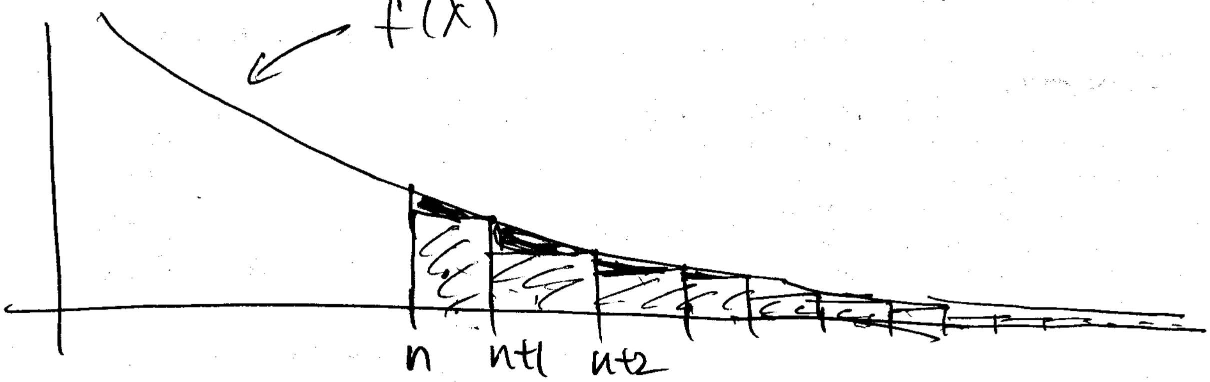
e.g. $\sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{5}\right)^k$

converges by A.S.T. but also since it is geometric with $|r| = \frac{1}{5} < 1$.

$$\sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k = \frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$$

$$\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4} = 1.25$$

8.4 35) $\sum_{k=1}^{\infty} \frac{1}{k^6}$ $\qquad a_k = f(k)$



$$R_n = \sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} f(x)\,dx$$

$$\sum_{k=1}^{\infty} \frac{1}{k^6} - \sum_{k=1}^{n} \frac{1}{k^6} = \sum_{k=n+1}^{\infty} \frac{1}{k^6} \le \int_n^{\infty} \frac{1}{x^6}\,dx$$

$$= \lim_{b\to\infty} \int_n^b x^{-6}\,dx = \lim_{b\to\infty} -\frac{1}{5}x^{-5}\Big|_n^b$$

$$= \lim_{b\to\infty} \left(\cancelto{0}{-\frac{1}{5}b^{-5}} + \frac{1}{5}n^{-5}\right) = \frac{1}{5n^5}$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k^6} \le \frac{1}{5n^5}$$

(b) For what n is $\frac{1}{5n^5} < 10^{-3}$

$$\frac{1}{5n^5} < \frac{1}{1000} \to 1000 < 5n^5$$

$$200 < n^5 \qquad \underline{\underline{n=3}}$$

Absolute convergence.

$$\sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{5}\right)^k \longrightarrow \text{converges by A.S.T.}$$

<u>or</u> ~~too~~ Forget the ± 1 part and look at

$$\sum_{k=0}^{\infty}\left|\left(-\frac{1}{5}\right)^k\right| = \sum_{k=0}^{\infty}\left|(-1)^k\left(\frac{1}{5}\right)^k\right| = \sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k$$

converges (geom. series)

We say: $\sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k$ converges <u>absolutely</u>.

We say $\sum_{k=1}^{\infty} a_k$ <u>converges</u> <u>absolutely</u>

if $\sum_{k=1}^{\infty}|a_k|$ converges.

<u>Thm</u>: If $\sum a_k$ converges absolutely then $\sum a_k$ converges.

Q: If $\sum a_k$ converges, does $\sum a_k$ converge absolutely? <u>NO</u>

$$\sum_{k=1}^{\infty}\underbrace{(-1)^{k+1}\frac{1}{k}}_{a_k} \text{ converges } \underline{\text{but}} \sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty}\underbrace{\frac{1}{k}}_{|a_k|} \text{ diverges.}$$

If $\sum a_k$ converges but $\sum |a_k|$ diverges then we say $\sum a_k$ <u>converges conditionally</u>

<u>e.g.</u> $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{k^2}$ ← Does this converge?

unpredictable mix of positive + negative terms.

Look at $\displaystyle\sum_{k=1}^{\infty} \left| \frac{\cos(k)}{k^2} \right| = \sum_{k=1}^{\infty} \frac{|\cos(k)|}{k^2}$ ← suspect convergence.

direct comparison:

$$\frac{|\cos(k)|}{k^2} \leq \frac{1}{k^2}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{|\cos k|}{k^2}$ converges by Direct Comp to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

so $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges absolutely hence it converges.

eg $\sum_{k=1}^{\infty} \frac{\cos(k)}{k}$ converges? diverges? We don't know!