Quiz 11 8.4

Ratio / Root / Comparison Tests

① Direct Comparison:

$\sum a_k$, $\sum b_k$, $a_k \geq 0$, $b_k \geq 0$

$0 \leq a_k \leq b_k$ for all k (or eventually for all k)

$\Longrightarrow$ $\sum b_k$ conv $\Rightarrow$ $\sum a_k$ conv.

$0 \leq b_k \leq a_k$: $\sum b_k$ div $\Rightarrow$ $\sum a_k$ diverges.

② Limit Comparison:

Suppose $\lim_{k\to\infty} \frac{a_k}{b_k} = L$

1) $0 < L < \infty \Longrightarrow \sum a_k, \sum b_k$ converge or diverge together

2) $L = 0 \Longrightarrow (\sum b_k \text{ conv} \Rightarrow \sum a_k \text{ conv.})$

3) $L = \infty \Longrightarrow (\sum b_k \text{ div} \Rightarrow \sum a_k \text{ div.})$

③ Ratio Test

$\sum a_k$, $a_k \geq 0$, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$

1) if $0 \leq r < 1$, $\sum a_k$ converges

2) if $r > 1$, $\sum a_k$ diverges

3) if $r = 1$, inconclusive

$$\boxed{\text{Idea: } \sum_{k=0}^{\infty} \underbrace{ar^k}_{a_k} \qquad \frac{a_{k+1}}{a_k} = \frac{ar^{k+1}}{ar^k} = r}$$

So saying $\frac{a_{k+1}}{a_k} \to r$ says that a_k behaves like r^k, so $\sum a_k$ behaves like $\sum_0^{\infty} r^k$.

e.g. What does "inconclusive" mean?

$\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges (p-series, $p = 3 > 1$).

$\frac{1}{k^3} \leftarrow a_k$

Apply Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{\frac{1}{(k+1)^3}}{\frac{1}{k^3}} = \frac{k^3}{(k+1)^3} \to 1$

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

a_k

Apply Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \to 1$

e.g. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

a_k

$$\boxed{Q! \quad \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k}$$

convergent geom series

Does the k^2 on top mess this up? NO

Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}} = \frac{2^k}{k^2} \cdot \frac{(k+1)^2}{2^{k+1}} = \frac{1}{2}\left(\frac{k+1}{k}\right)^2$$

$$\to \frac{1}{2} < 1 \quad \therefore \sum_{k=1}^{\infty} \frac{k^2}{2^k} \text{ converges.}$$

$$\boxed{\frac{2^k}{2^{k+1}} = \frac{\cancel{2^k}}{\cancel{2^k} \cdot 2^1} = \frac{1}{2}}$$

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{k^6}{k!}$ ← converges

Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{\dfrac{(k+1)^6}{(k+1)!}}{\dfrac{k^6}{k!}} =$$

$$= \frac{(k+1)^6}{(k+1)!} \cdot \frac{k!}{k^6} = \frac{k!}{(k+1)!} \cdot \frac{(k+1)^6}{k^6}$$

$$= \frac{\cancel{k(k-1)(k-2)\cdots(2)(1)}}{(k+1)\cancel{(k)(k-1)\cdots(2)(1)}} \left(\frac{k+1}{k}\right)^6 = \left(\frac{1}{k+1}\right) \cdot \left(\frac{k+1}{k}\right)^6$$

$$\left(\frac{k+1}{k}\right)^6 \qquad 0 \qquad 1$$

$$\longrightarrow 0 < 1$$

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ Ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}$$

$$= \frac{(k+1)!}{k!} \cdot \frac{k^k}{(k+1)^{k+1}} = (k+1) \cdot \frac{k^k}{(k+1)^{k+1}} = \frac{k^k}{(k+1)^k}$$

$$\frac{(k+1)!}{k!} \to k+1$$

$$= \left(\frac{k}{k+1}\right)^k \leftarrow \text{indeterminate}$$

so L'Hopital will work.

But wait! $\left(\frac{k}{k+1}\right)^k = \frac{1}{\left(\frac{k+1}{k}\right)^k} = \frac{1}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{\to e}}$

$\longrightarrow \frac{1}{e} < 1$

$\therefore \sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges.

④ Root Test

$\sum a_k$, $a_k \geq 0$ $\lim_{k\to\infty} (a_k)^{1/k} = r$

1) $0 \leq r < 1$, $\sum a_k$ converges

2) $r > 1$, $\sum a_k$ diverges

3) $r = 1$, inconclusive

eg $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ Root Test $\left(\frac{k^2}{2^k}\right)^{1/k} = \frac{k^{2/k}}{2}$

$k^{2/k} \longrightarrow ?$ $k^{2/k} = \left(\underbrace{k^{1/k}}_{\to 1}\right)^2 \longrightarrow 1$ as $k \to \infty$.

So $\left(\frac{k^2}{2^k}\right)^{1/k} \longrightarrow \frac{1}{2} < 1$ so $\sum \frac{k^2}{2^k}$ conv.

20) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k+1}{2k}\right)^{k}$ $\qquad (a_k)^{1/k} = \left[\left(\frac{k+1}{2k}\right)^{k}\right]^{1/k}$

$$= \frac{k+1}{2k} \longrightarrow \frac{1}{2} < 1$$

$\displaystyle\sum_{1}^{\infty}\left(\frac{k+1}{2k}\right)^{k}$ converges.

$$\sum_{k=1}^{\infty} a_k = S \qquad \sum_{k=1}^{n} a_k = S_n$$

What is an upperbound on $S-S_n$?

$$\int_{n+1}^{\infty} f(x)\,dx \leq \underbrace{S-S_n}_{R_n} \leq \int_{n}^{\infty} f(x)\,dx$$

$$\sum_{k=1}^{\infty} \frac{3}{2^k} \longrightarrow R_n \leq \int_{n}^{\infty} \frac{3}{2^x}\,dx$$

$$= \lim_{b \to \infty} \int_{n}^{b} \frac{3}{2^x}\,dx$$