Quiz 11 - 8.4

$\sum_{k=1}^{\infty} a_k$ Q: Does the series converge? Yes/No

(Evaluating the series is hard.)

Convergence Tests:

① Divergence test: If $\lim_{k\to\infty} a_k \neq 0$, $\sum_{1}^{\infty} a_k$ diverges

② Integral test: if $a_k = f(k)$ where (i) f is decreasing (ii) continuous (iii) positive, then $\sum a_k$ diverges or converges with $\int_1^{\infty} f(x)dx$.

③ $\sum_{k=0}^{\infty} a \cdot r^k \to$ converges if $|r|<1$
diverges if $|r| \geq 1$.

$\sum_{k=0}^{\infty} \frac{1}{n^p} \longrightarrow$ converges if $p>1$
$(p>0)$ diverges if $0<p\leq 1$

8.5 Ratio, Root, Comparison Tests.

① Comparison Test

If $\sum a_k$, $\sum b_k$ have positive terms then 1) if $0<a_k \leq b_k$ for all k and $\sum b_k$ converges then $\sum a_k$ converges.

2) if $0 < b_k \leq a_k$ for all k and $\sum b_k$ diverges then $\sum a_k$ diverges.

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1}$

Think: $\dfrac{k}{k^3+1} \sim \dfrac{k}{k^3} = \dfrac{1}{k^2}$

if k large.

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(2>1)$

So we want

$\dfrac{k}{k^3+1} \overset{?}{\leq} \dfrac{1}{k^2}$

$k^3 \overset{?}{\leq} k^3+1$ ✓

So verified.

Since $\dfrac{k}{k^3+1} \leq \dfrac{1}{k^2}$ all k

$\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1}$ converges since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

e.g. $\displaystyle\sum_{k=1}^{\infty} \sqrt{\frac{k}{k^3+1}}$

$\sqrt{\dfrac{k}{k^3+1}} \sim \sqrt{\dfrac{1}{k^2}} = \dfrac{1}{k}$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges so we suspect $\displaystyle\sum_{1}^{\infty} \sqrt{\frac{k}{k^3+1}}$ diverges

Want:

$\dfrac{1}{k} \overset{?}{\leq} \sqrt{\dfrac{k}{k^3+1}}$

$\dfrac{1}{k^2} \overset{?}{\leq} \dfrac{k}{k^3+1}$

$k^3+1 \overset{?}{\leq} k^3$

$1 \overset{?}{\leq} 0$ NO

Try: $\dfrac{1}{2k} \overset{?}{\leq} \sqrt{\dfrac{k}{k^3+1}}$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

$\dfrac{1}{4k^2} \overset{?}{\leq} \dfrac{k}{k^3+1}$

$k^3+1 \overset{?}{\leq} 4k^3$

$-k^3 \qquad -k^3$

$1 \overset{?}{\leq} 3k^3$ ✓

verified

$$\boxed{\text{Since } \sum_1^{\infty} \frac{1}{2k} \text{ diverges so does } \sum_1^{\infty}\sqrt{\frac{k}{k^3+1}}.}$$

e.g. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{[k\ln(k)]^2} = \sum_{k=2}^{\infty}\frac{1}{k^2(\ln k)^2}$

Want:

$\dfrac{1}{k^2(\ln k)^2} \overset{?}{\leq} \dfrac{1}{k^2}$

$k^2 \overset{?}{\leq} k^2(\ln k)^2$

$1 \overset{?}{\leq} (\ln k)^2$

$1 \overset{?}{\leq} \ln k$

true if $k \geq 3$

verified

Think: $\dfrac{1}{k^2(\ln k)^2}$ goes to 0 faster than $\dfrac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges. So suspect $\sum \dfrac{1}{k^2(\ln k)^2}$ converges.

$$\boxed{\text{Since } \sum \frac{1}{k^2} \text{ converges so does } \sum \frac{1}{k^2(\ln k)^2}}$$

e.g 30) $\sum_{k=1}^{\infty} \frac{.0001}{k+4} = .0001 \sum_{k=1}^{\infty} \frac{1}{k+4}$

Just look at $\sum_{k=1}^{\infty} \frac{1}{k+4}$. $\frac{1}{k+4} \sim \frac{1}{k}$

Want $\frac{1}{k} \overset{?}{\leq} \frac{1}{k+4}$ NO   Suspect $\sum_{1}^{\infty} \frac{1}{k+1}$ diverges because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

How about $\frac{1}{2k} \overset{?}{\leq} \frac{1}{k+4}$

$k+4 \overset{?}{\leq} 2k$

$4 \overset{?}{\leq} k$ ✓

verified for $k \geq 4$.

$\boxed{\sum_{k=1}^{\infty} \frac{1}{k+4} \text{ diverges since } \sum_{k=1}^{\infty} \frac{1}{2k} \text{ diverges.}}$

34) $\sum_{k=1}^{\infty} \frac{1}{3^k - 2^k}$ $\frac{1}{3^k - 2^k} \sim \frac{1}{3^k} = \left(\frac{1}{3}\right)^k$

Want $\frac{1}{3^k - 2^k} \overset{?}{\leq} \frac{1}{3^k}$ NO   Suspect $\sum \frac{1}{3^k - 2^k}$ conv. since $\sum \frac{1}{3^k}$ conv.

Try: $\frac{1}{2(3^k - 2^k)} \overset{?}{\leq} \frac{1}{3^k}$ $2^{k+1} \overset{?}{\leq} 3^k$ ✓ VERIFIED.

$3^k \overset{?}{\leq} 2 \cdot 3^k - 2^{k+1}$ $(k+1)\ln(2) \leq k \ln(3)$

$0 \overset{?}{\leq} 3^k - 2^{k+1}$ $4 \quad \ln 2 \leq k(\cancel{\ln 3} - \cancel{\ln 2})$ ✓

So $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2(3^k-2^k)}$ converges since $\displaystyle\sum_{1}^{\infty} \frac{1}{3^k}$ converges.

So $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k-2^k}$ converges.

$$\sum_{k=1}^{\infty} \frac{1}{3^k-2^k} \qquad \frac{1}{3^k-2^k} = \frac{1}{2^k\left(\left(\frac{3}{2}\right)^k-1\right)}$$

want: $\qquad$ comp to $\dfrac{1}{2^k}$

$$\frac{1}{3^k-2^k} = \frac{1}{2^k\left(\left(\frac{3}{2}\right)^k-1\right)} \overset{?}{\le} \frac{1}{2^k}$$

$$2^k \overset{?}{\le} 2^k\left(\left(\frac{3}{2}\right)^k-1\right)$$

$$1 \overset{?}{\le} \left(\frac{3}{2}\right)^k-1$$

$$2 \overset{?}{\le} \left(\frac{3}{2}\right)^k \checkmark$$

verified if $k \ge 2$.

② Limit Comparison Test.

$\sum a_k$, $\sum b_k$ positive terms.

Suppose $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L$

1) If $0 < L < \infty$ then $\sum a_k$ and $\sum b_k$ converge or diverge together

2) If $L = 0$ then $a_k \to 0$ faster than b_k.
so if $\sum b_k$ converges, so does $\sum a_k$

3) If $L = \infty$ then $b_k \to 0$ faster than a_k
so if $\sum b_k$ diverges so does $\sum a_k$.

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1}$ $\qquad \dfrac{k}{k^3+1} \sim \dfrac{1}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{k}{k^3+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1$$

$\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^3+1}$ converges.

e.g. $\displaystyle\sum_{k=1}^{\infty}\sqrt{\frac{k}{k^3+1}}$ $\qquad \displaystyle\sqrt{\frac{k}{k^3+1}} \sim \frac{1}{k}$

$$\lim_{k\to\infty}\frac{\sqrt{\frac{k}{k^3+1}}}{\frac{1}{k}} = \lim_{k\to\infty}\sqrt{\frac{\frac{k}{k^3+1}}{\frac{1}{k^2}}} = \lim_{k\to\infty}\sqrt{\frac{k^3}{k^3+1}} = 1$$

$\displaystyle\sum_{k=1}^{\infty}\sqrt{\frac{k}{k^3+1}}$ diverges with $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$

eg $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3^k-2^k}$ $\qquad \displaystyle\frac{1}{3^k-2^k} \sim \frac{1}{3^k} = \left(\frac{1}{3}\right)^k$

$$\lim_{k\to\infty}\frac{\frac{1}{3^k-2^k}}{\frac{1}{3^k}} = \lim_{k\to\infty}\frac{3^k}{3^k-2^k} = \lim_{k\to\infty}\frac{1}{1-\underbrace{\left(\frac{2}{3}\right)^k}_{0}} = 1$$

$\displaystyle\sum_{1}^{\infty}\frac{1}{3^k-2^k}$ converges with $\displaystyle\sum_{1}^{\infty}\left(\frac{1}{3}\right)^k$