Quiz 10 8.3 only

Integral Test.

$$\sum_{k=1}^{\infty} a_k \ , \quad a_k = f(k), \ f \text{ is } \begin{array}{l} 1.\ \text{continuous} \\ 2.\ \text{positive.} \\ 3.\ \text{decreasing} \end{array}$$

$\sum_{1}^{\infty} a_k$ converges or diverges with $\int_1^{\infty} f(x)\,dx$.

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(10) $$\sum_{k=1}^{\infty}\left[2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right] \qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

$$= \sum_{k=1}^{\infty} 2\left(\frac{3}{5}\right)^k + \sum_{k=1}^{\infty} 3\left(\frac{4}{9}\right)^k$$

$$2\left(\frac{3}{5} + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^3 + \cdots\right) = 2\cdot\frac{3}{5}\left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)$$

$$\sum_{k=1}^{\infty} 2\left(\frac{3}{5}\right)^k = 2\cdot\frac{3}{5}\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k$$

$$= 2\cdot\frac{3}{5}\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k + 3\cdot\frac{4}{9}\sum_{k=0}^{\infty}\left(\frac{4}{9}\right)^k$$

$$= \frac{6}{5}\,\frac{1}{1-\frac{3}{5}} + \frac{4}{3}\cdot\frac{1}{1-\frac{4}{9}} = \frac{6}{5-3} + \frac{4}{3-\frac{4}{3}}\cdot\frac{3}{3}$$

$$= 3 + \frac{12}{5} = \frac{27}{5} \ //$$

14) $$\sum_{k=0}^{\infty} \frac{2-3^k}{6^k} = \sum_{k=0}^{\infty}\left(\frac{2}{6^k} - \frac{3^k}{6^k}\right)$$

$2\left(\frac{1}{6}\right)^k$ $\quad$ $\left(\frac{3}{6}\right)^k = \left(\frac{1}{2}\right)^k$

$$= \sum_{k=0}^{\infty} 2\left(\frac{1}{6}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k$$

$$= \frac{2}{1-\frac{1}{6}} - \frac{1}{1-\frac{1}{2}} = \frac{2}{\frac{5}{6}} - \frac{1}{\frac{1}{2}} = \frac{12}{5} - 2 = \frac{2}{5}$$ //.

Back to Integral Test, ...

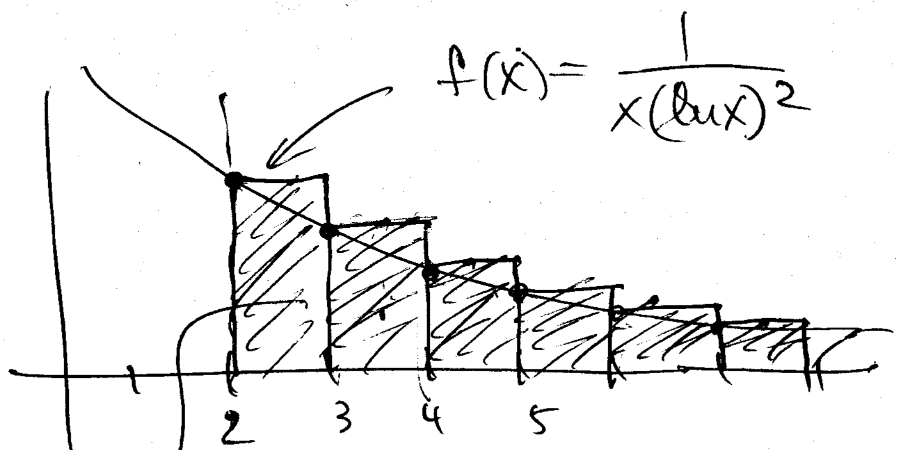
28) $$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \qquad f(x) = \frac{1}{x(\ln x)^2} \quad x \in [2,\infty). \ (x \geq 2).$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$$

$$\boxed{\int \frac{1}{x(\ln x)^2}\,dx \begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix} = \int \frac{1}{u^2}\,du = \frac{-1}{u} = \frac{-1}{\ln x}}$$

$$= \lim_{b\to\infty}\left(\frac{-1}{\ln x}\Big|_2^b\right) = \lim_{b\to\infty}\left(\overset{0}{\cancel{\frac{-1}{\ln b}}} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

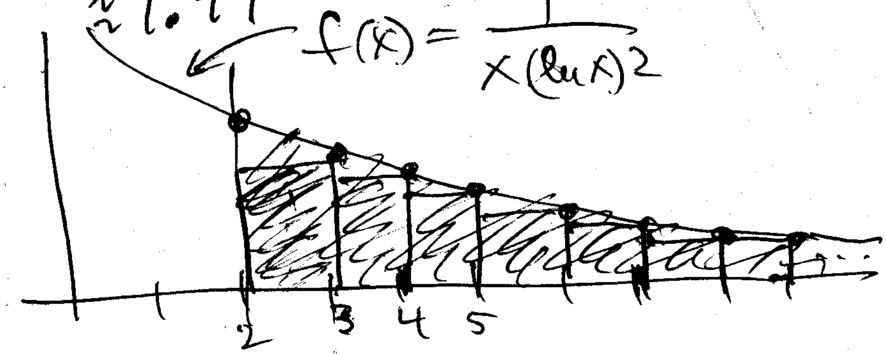
$\therefore \displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.



area under curve $= \frac{1}{\ln 2}$

area under rectangles $= \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

$$\text{so } \underbrace{\frac{1}{\ln 2}}_{\approx 1.44} \leq \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \leq \underbrace{\frac{1}{\ln 2} + \frac{1}{2(\ln 2)^2}}_{\approx 2.48}$$



area under rectangles $= \sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2} \leq \frac{1}{\ln 2}$$

Q: What about a partial sum?

For $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$, how good is the approx using 10 terms? For this we need:

$$\int_2^{n+1} \frac{1}{x(\ln x)^2}\,dx \leq \sum_{k=2}^{n} \frac{1}{k(\ln k)^2} \leq \int_2^{n} \frac{1}{x(\ln x)^2}\,dx + \frac{1}{2(\ln 2)^2}$$

So
$$\int_2^{12} \frac{1}{x(\ln x)^2}\,dx \leq \underbrace{\sum_{k=2}^{11} \frac{1}{k(\ln k)^2}}_{S_{10}} \leq \int_2^{11} \frac{1}{x(\ln x)^2} + \frac{1}{2(\ln 2)^2}$$

But I want. $\left(\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} - \sum_{k=2}^{11} \frac{1}{k(\ln k)^2} \right)$

$$= \sum_{k=12}^{\infty} \frac{1}{k(\ln k)^2}$$

$$\left(\underbrace{\sum_{k=12}^{\infty} \frac{1}{k(\ln k)^2}}_{\text{error}} \leq \int_{12}^{\infty} \frac{1}{x(\ln x)^2}\,dx \right)$$

$$= \lim_{b\to\infty} \int_{12}^{b} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{\ln x}\right|_{12}^{b}$$

$$= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 12}\right) = \frac{1}{\ln 12} \approx .40$$

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} = \left(\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \cdots + \frac{1}{11(\ln 11)^2}\right)$$

$$\underbrace{\left(+\frac{1}{12(\ln 12)^2} + \frac{1}{13(\ln 13)^2} + \cdots\right)}$$

How big is this error?

$$\sum_{k=12}^{\infty} \frac{1}{k(\ln k)^2}$$ less than .40

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e.g. Look at $\sum_{k=1}^{\infty} \frac{1}{k}$. How fast does it diverge?

How many terms to get past 10?

Integral test: $\int_1^{\infty} \frac{1}{x}\,dx = \ln x\Big|_1^{\infty} = \infty$.

$$\int_1^{n+1} \frac{1}{x}\,dx \le \sum_{k=1}^{n} \frac{1}{k} \le \int_1^{n} \frac{1}{x}\,dx + 1$$

$$\ln(n+1) \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \ln(n)$$

For $\sum_{k=1}^{n} \frac{1}{k} \ge 10$ I will need at least $1 + \ln n \ge 10$.

$\ln n \ge 9 \quad n \ge e^9 = 8103$

For $\sum_{k=1}^{n} \frac{1}{k} \ge 100$, $\quad 1 + \ln n \ge 100$

$\ln n \ge 99$

$n \ge e^{99} > 9 \times 10^{42}$

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p-series.

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. What about $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

converges!

Look at $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$

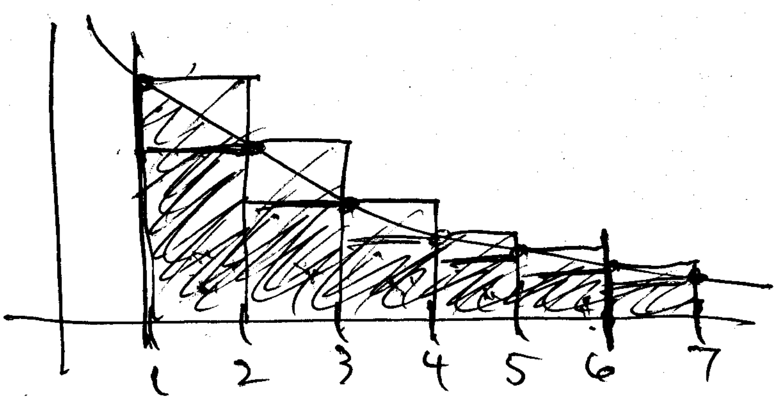
$= \lim_{b\to\infty}\left(-\frac{1}{b} + 1\right) = 1$

$\sum_{k=1}^{\infty} \frac{1}{k^p} \quad (p>0)$ $\qquad \int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } 0<p\le 1 \end{cases}$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } 0<p\le 1 \end{cases}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^{2/3}} \text{ diverges}, \quad \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \text{ converges, etc.}$$

$$\int_1^{n+1} \frac{1}{x}\,dx \le \sum_{k=1}^{n} \frac{1}{k} \le \int_1^{n} \frac{1}{x}\,dx + 1$$



$$\int_1^7 f(x)\,dx \le \sum_{k=1}^{6} a_k \le \int_1^6 f(x)\,dx + a_1$$

$$\sum_{k=2}^{6} a_k \le \int_1^6 f(x)\,dx$$