Exam 2 handed back Monday.
Midterm grades available by email
Quiz 9 - 8.1, 8.2 (basic questions on sequences and series).

Series : $\sum_{k=1}^{\infty} a_k$ → converge ← $\lim_{n\to\infty} S_n$ exists, $S_n = \sum_{k=1}^{n} a_k$ partial sums

→ diverge ← $\lim_{n\to\infty} S_n$ does not exist.

If $\sum_{k=1}^{\infty} a_k$ converges, we write $\sum_{k=1}^{\infty} a_k = L$

where $L = \lim_{n\to\infty} S_n$.

Seen: (a) $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ so diverges

(b) Another divergent series

$$\sum_{k=1}^{\infty} (-1)^{k+1} = (1(-1+1)(-1+1)(-1+1)-1)+\cdots$$

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} : \{1, 0, 1, 0, 1, 0, 1, 0, \dots\}$$

S_n never converges so $\sum_{k=1}^{\infty} (-1)^{k+1}$ diverges

Usually hard to actually find the sum of a series. In a few cases we can.

① Geometric series

$$\sum_{k=0}^{\infty} a r^k \text{ for } a, r \text{ fixed numbers.}$$

Find formula for $S_n = \sum_{k=0}^{n} a r^k$.

Fact: $\displaystyle\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + r^3 + \cdots + r^n$$

$$r\left(\sum_{k=0}^{n} r^k\right) = r\left(1 + r + r^2 + r^3 + \cdots + r^n\right)$$
$$= r + r^2 + r^3 + r^4 + \cdots + r^{n+1}$$

$$\left(\sum_{k=0}^{n} r^k\right) - r\left(\sum_{k=0}^{n} r^k\right) = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^n}$$
$$- \left(\cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^n} + r^{n+1}\right)$$
$$= 1 - r^{n+1}$$

$$\downarrow$$

$$(1-r)\sum_{k=0}^{n} r^k = 1 - r^{n+1} \longrightarrow \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{k=0}^{\infty} ar^k = a\sum_{k=0}^{\infty} r^k = a\cdot\lim_{n\to\infty}\left(\frac{1-r^{n+1}}{1-r}\right)$$

$$= \frac{a}{1-r}\lim_{n\to\infty}(1-r^{n+1}).$$

If $r>1 \Rightarrow (r^{n+1} \to \infty$ as $n\to\infty)$

If $0\leq r<1 \Rightarrow r^{n+1} \to 0$ as $n\to\infty$.

If $-1<r<0 \Rightarrow r^{n+1} \to 0$ as $n\to\infty$.

If $r<-1 \Rightarrow r^{n+1}$ diverges as $n\to\infty$.

If $|r|<1$ then $\lim_{n\to\infty}(1-r^{n+1}) = 1$

So $\boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}} = a\cdot\frac{1}{1-r}$

If $|r|\geq 1$ then $\lim_{n\to\infty}(1-r^{n+1})$ diverges.

So $\boxed{\sum_{k=0}^{\infty} ar^k \text{ diverges}}$

If $r = 1$ or -1

$$\sum_{k=0}^{\infty} ar^k = a + a + a + a + a + \cdots \text{ diverges}$$

$$\sum_{k=0}^{\infty} ar^k = a - a + a - a + a - a + \cdots \text{ diverges.}$$

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eg. 8) $\sum_{k=0}^{10} \left(\frac{1}{4}\right)^k = \frac{1-\left(\frac{1}{4}\right)^{11}}{1-\frac{1}{4}} = \frac{1-\left(\frac{1}{4}\right)^{11}}{\frac{3}{4}}$

$$= \frac{4}{3}\left(1-\left(\frac{1}{4}\right)^{11}\right)$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$

10) $\sum_{k=4}^{12} 2^k = 2^4 + 2^5 + 2^6 + \cdots + 2^{12}$

$$= 2^4\left(1 + 2 + 4 + \cdots + 2^8\right)$$

$$= 2^4 \sum_{k=0}^{8} 2^k = 16\left[\frac{1-2^9}{1-2}\right] = 16\left[\frac{2^9-1}{2-1}\right]$$

$$= 16\left(2^9-1\right) = 16(511) = 8176$$

18) $\frac{1}{3}+\frac{1}{5}+\frac{3}{25}+\frac{9}{125}+\cdots+\frac{243}{15625}$

$$=\frac{1}{3}\left(1+\frac{3}{5}+\frac{9}{25}+\frac{27}{125}+\cdots+\frac{729}{15625}\right)$$

$$=\frac{1}{3}\left(\left(\frac{3}{5}\right)^0+\left(\frac{3}{5}\right)^1+\left(\frac{3}{5}\right)^2+\left(\frac{3}{5}\right)^3+\cdots+\left(\frac{3}{5}\right)^6\right)$$

$$=\frac{1}{3}\sum_{k=0}^{6}\left(\frac{3}{5}\right)^k=\frac{1}{3}\cdot\frac{1-\left(\frac{3}{5}\right)^7}{1-\frac{3}{5}}=\frac{1}{3}\left(1-\left(\frac{3}{5}\right)^7\right)\cdot\frac{5}{2}$$

$$=\frac{5}{6}\left(1-\left(\frac{3}{5}\right)^7\right).$$

~~20)~~

26) $\sum_{n=2}^{\infty}\frac{5}{2^n}=\sum_{n=2}^{\infty}5\cdot\left(\frac{1}{2}\right)^n$

$$=5\left(\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots\right)$$

$$=5\left(\frac{1}{2}\right)^2\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\right]$$

$$=\underset{\substack{\uparrow\\ a}}{\frac{5}{4}}\sum_{n=0}^{\infty}\underset{\substack{\uparrow\\ r}}{\left(\frac{1}{2}\right)}^n=\frac{5}{4}\cdot\frac{1}{1-\frac{1}{2}}=\frac{5}{2}\ //.$$

46) $5.12\overline{83}8383\cdots$

$$= 5.12 + .00838383\cdots$$

$$= 5.12 + \frac{1}{100}(.838383\cdots)$$

$$.83 + .0083 + .000083 + \cdots$$

$$= .83\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$$

$$= .83\left(1 + .01 + (.01)^2 + (.01)^3 + \cdots\right)$$

$$= .83 \sum_{k=0}^{\infty} (.01)^k = .83 \cdot \frac{1}{1-.01} = \frac{.83}{.99}$$

$$= \frac{83}{99}$$

$$= \frac{512}{100} + \frac{83}{9900} = \frac{50771}{9900}$$

1. $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots$

To get from one term to next multiply by r.

In other words $\frac{r^{k+1}}{r^k} = r$

so ratio of successive terms is constant,

$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

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(2) Telescoping series.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} = \frac{1}{k} - \frac{1}{k+1}$$

$1 = A(k+1) + Bk$

$k=0$: $A = 1$

$k=-1$: $1 = -B$

$B = -1$

look at $S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 + \left(\cancel{-\frac{1}{2} + \frac{1}{2}}\right) + \left(\cancel{-\frac{1}{3} + \frac{1}{3}}\right) + \left(\cancel{-\frac{1}{4} + \frac{1}{4}}\right) + \cdots + \left(\cancel{-\frac{1}{n} + \frac{1}{n}}\right) - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

So in this case $S_n = 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1$ so $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

52) $\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ divergent.

$$S_n = \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) = (\cancel{\sqrt{2}} - 1) + (\cancel{\sqrt{3}} - \cancel{\sqrt{2}}) + (\cancel{\sqrt{4}} - \cancel{\sqrt{3}}) + (\cancel{\sqrt{5}} - \cancel{\sqrt{4}}) + \cancel{\cdots} + (\sqrt{n+1} - \cancel{\sqrt{n}})$$

$$= \sqrt{n+1} - 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1} - 1) = \infty$$

<u>Remark</u>: Having a formula for S_n is not common.

Usually best we can do is decide if a series converges or diverges without finding its sum.