Sequences + Series

Sequence: $\{a_n\} = \{a_1, a_2, a_3, a_4, \dots\}$

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

Two sequences associated to a series:

① Terms: $a_1, a_2, a_3, a_4, \dots$

② Partial sums: $a_1, a_1+a_2, a_1+a_2+a_3, a_1+a_2+a_3+a_4,$

$\dots$

Better notation: $S_n = \sum_{k=1}^{n} a_k$

$= a_1 + a_2 + \cdots + a_n.$

Sequences. Want: $\lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} S_n$

① $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ if $f(n) = a_n$.

②

② Relative rates of growth.

$\{n^n\}_{n=1}^{\infty} = \{1, 4, 27, 256, 3125, \ldots\}$

$\{n!\}_{n=1}^{\infty} = \{1, 2, 6, 24, 120, 720, \ldots\}$

$\{r^n\}_{n=1}^{\infty}$ $(r>1)$ $\underline{\underline{r=2}}$ $\{2, 4, 8, 16, 32, 64, 128, \ldots\}$

$\{n^p\}_{n=1}^{\infty}$ $p>0$ $\underline{p=2}$ $\{1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$

$\{\log_b(n)\}_{n=1}^{\infty}$ $\underline{b=2}$ $\{0, 1, 1.6, 2, 2.3, 2.6, 2.8, 3, 3.1, \ldots\}$

Infinite Series. $\sum_{n=1}^{\infty} a_n$

Convergence means $\lim_{n\to\infty} S_n$ <u>exists</u>.

<u>Sometimes</u> we have a formula for S_n and we can find $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$.

<u>Usually</u> we can't sum a series; we just try to decide if it converges, i.e. if sum is finite.

e.g. Here is a <u>divergent</u> series, i.e. it sums to infinity.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Here is why it sums to ∞.

$$\cancel{\underbrace{(1)}_{1} + \underbrace{\left(\frac{1}{2} + \frac{1}{3}\right)}_{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right)}_{4} + \underbrace{\left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15}\right)}_{8} + \frac{1}{16} + \cdots}$$

$$\cancel{2}\ \underbrace{\left(1 + \frac{1}{2}\right)}_{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{4} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{8} + \underbrace{\left(\frac{1}{17} + \cdots + \frac{1}{32}\right)}_{16} + \cdots$$

$$\geq \frac{3}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{4} + \underbrace{\left(\frac{1}{16} + \cdots + \frac{1}{16}\right)}_{8} + \underbrace{\left(\frac{1}{32} + \cdots + \frac{1}{32}\right)}_{16} + \cdots$$

$$= \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

<u>Fact:</u> $S_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln(n)$