Exam 2 - Monday 3-25 7.1-7.4, 7.7
No calculators, 1 3x5 card with notes.
Quiz 8 - Thursday 7.6, 7.7.
Oval Review sign-up sheet is on line <u>now</u>.

Sequences and Series (Idea)

Polynomials: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
$$= \sum_{k=0}^{n} a_k x^k$$

<u>Notice</u>: $p(0) = a_0 = 0!\,a_0$

$p'(0) = a_1 = 1!\,a_1$

$p'(x) = 0 + a_1 + 2a_2 x + \overset{3a_3x^2}{\cdots} + n a_n x^{n-1}$

$p''(0) = 2a_2 = (2\cdot 1)a_2 = 2!\,a_2$

$p''(x) = 0 + 0 + 2a_2 + 6a_3 x + \cdots + n(n-1)a_n x^{n-2}$

$p'''(0) = 6a_3 = (3\cdot 2\cdot 1)a_3 = 3!\,a_3$

$p^{(4)}(0) = 24a_4 = (4\cdot 3\cdot 2\cdot 1)a_4 = 4!\,a_4$

In general $p^{(k)}(0) = k!\,a_k$ so $\boxed{a_k = \frac{1}{k!}p^{(k)}(0)}$

$$\boxed{p(x) = \sum_{k=0}^{n} \frac{1}{k!} p^{(k)}(0)\, x^k}$$

Q: What if I replace $p(x)$ by an arbitrary function $f(x)$? e.g. $f(x) = \sin(x)$

$f(x) = e^x$ $\quad f(x) = \ln(x)$

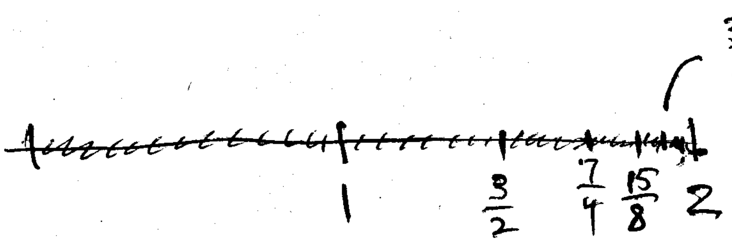
Can I write $\underbrace{f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\, x^k}_{\text{Taylor series}}$?

First we have to make sense out of infinite sums, e.g.

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots$$

Observation: $\sum_{k=0}^{\infty} a_k$ need not be infinite.

e.g. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 2$



$1;\ 1 + \frac{1}{2} = \frac{3}{2},\ 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8},\ \frac{31}{16},\ \frac{63}{32}, \ldots$

Q: When is $\sum_{k=0}^{\infty} a_k$ finite?

8.1/8.2 Sequences.

A sequence is a function whose domain is $\mathbb{N} = \{1, 2, 3, \dots\}$ or $\{0, 1, 2, 3, \dots\}$.

Instead of $f(k)$, we write a_k.

Think of a sequence as a list of numbers

$$\{a_k\} = \{a_1, a_2, a_3, a_4, \dots\}$$

e.g. If $f(k) = k^2$ write $a_k = k^2$ so

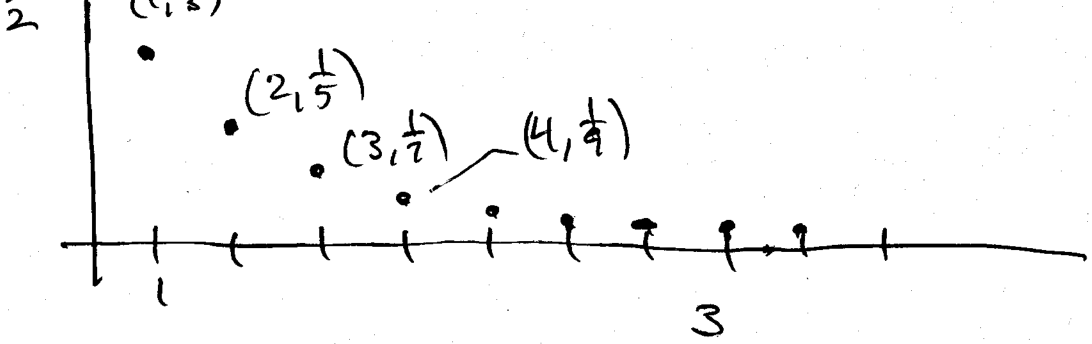
$$\{1, 4, 9, 16, 25, 36, \dots\}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 & a_4 \end{array} \quad \dots$$

e.g. $a_k = \dfrac{1}{2k+1}$

$$\{a_1, a_2, a_3, \dots\} = \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \dots\right\}$$

We can sometimes graph a sequence:



e.g. Sequences can be defined recursively:

- $a_1 = -2 \qquad a_{n+1} = \dfrac{n\, a_n}{n+1}$

$$a_2 = \frac{1 \cdot a_1}{1+1} = \frac{-2}{2} = -1$$

$$a_3 = \frac{2 \cdot a_2}{2+1} = \frac{-2}{3} = -\frac{2}{3}$$

$$a_4 = \frac{3 \cdot a_3}{3+1} = \frac{-2}{4} = -\frac{1}{2}$$

$$a_5 = \frac{4 \cdot a_4}{4+1} = \frac{-2}{5} = -\frac{2}{5}$$

$\vdots$

e.g. Fibonacci sequence:

$a_1 = 1 \qquad a_{n+1} = a_n + a_{n-1}$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

$\vdots$

Main Question:
Given a sequence a_k
what happens to a_k as $k \to \infty$?
We write: What is $\lim_{k\to\infty} a_k$?

A series is the sum of a sequence.

We write $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

e.g. $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

$= 1$

Look at partial sums.

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n = \frac{2^n - 1}{2^n}$$

We say $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1$

Sequence of terms $= \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots \right\}$

Partial sums

$\frac{1}{2}$

$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$

$\vdots$

$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Sequence of partial sums $= \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots \right\}$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

A series has associated to it $\underline{\underline{2}}$ sequences: $\displaystyle\sum_{k=1}^{\infty} a_k$

sequence of terms: $\{a_1, a_2, a_3, \cdots\}$

sequence of partial sums: $\{S_1, S_2, S_3, S_4, \cdots\}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Want to look at $\lim_{k\to\infty} a_k$.

To find limits of sequences use same ideas and techniques as with $\lim_{x\to\infty} f(x)$.

If we can write $a_k = f(k)$ for some $f(x)$ then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{k\to\infty} a_k = L.$$

eg $\displaystyle\lim_{n\to\infty} \frac{n^4}{3n^4+1} = \lim_{x\to\infty} \frac{x^4}{3x^4+1} = \frac{1}{3}$

e.g. $$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = 0$$

e.g. $$\lim_{n\to\infty} \sin(n\pi) = \lim_{n\to\infty} 0 = 0$$

$\sin(\pi) = 0$

$\sin(2\pi) = 0$

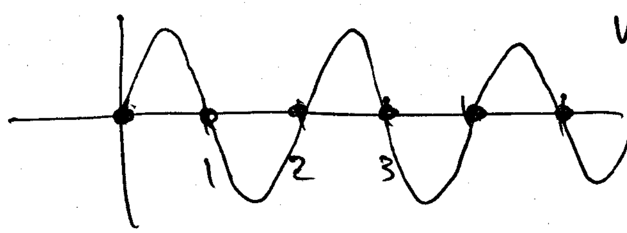
$\sin(3\pi) = 0$

$\vdots$

$\sin(n\pi) = 0$

$\lim_{x\to\infty} \sin(\pi x)$ Does not exist

In this case, replacing n by x does not work.



$\sin(\pi x)$

e.g. $$\lim_{n\to\infty} \frac{n^2}{n!}$$

$$\left\{1, 2, \frac{3}{2}, \frac{2}{3}, \frac{5}{24}, \frac{1}{20}, \cdots\right\}$$

$\frac{\cancel{2}\cancel{5}5}{\cancel{5}\cdot 4\cdot 3\cdot 2}$ $\frac{\cancel{3}\cancel{6}6}{6\cdot 5\cdot 4\cdot \cancel{3}\cdot 2}$

$$\lim_{x\to\infty} \frac{x^2}{x!}$$ ← Does not make sense

$x!$ ← undefined

Also in this case, replacing n by x does not work.

e.g. $\lim_{n\to\infty} r^n$ (r is a fixed real number)

$\lim_{n\to\infty} 2^n = \infty$ ($\lim_{x\to\infty} 2^x = \infty$)

$\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$ ($\lim_{x\to\infty} \left(\frac{1}{3}\right)^x = 0$)

$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ (looking at $\lim_{x\to\infty} \left(-\frac{1}{2}\right)^x$ doesn't work)

$\left\{ -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, \cdots \right\}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \infty & \text{if } |r|>1 \\ 1 & \text{if } r=1 \\ \text{Does not exist.} & \text{if } r=-1 \end{cases}$$

$(-1)^n : \{-1, 1, -1, 1, -1, 1, \cdots\}$