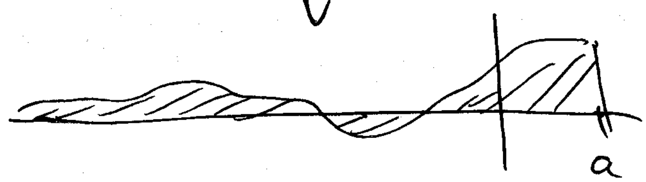
No office hrs today

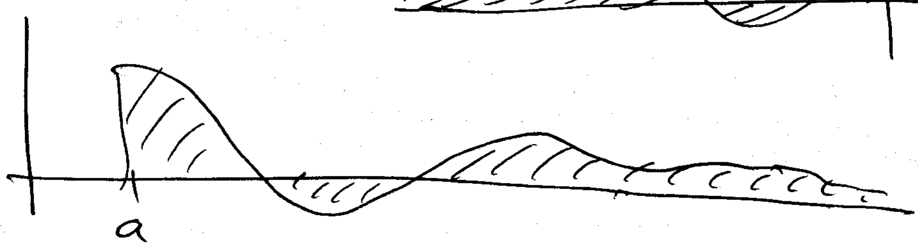
Quiz 8 - 7.6, 7.7 (calculators allowed)

Exam 2 - Monday 3/25 Ch. 7 (omit 7.5, 7.8).

Improper integrals

$\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^a f(x)\,dx$





An infinite or unbounded region can have finite area, but it does not have to.

In short $\int_a^\infty f(x)\,dx$ can $= \infty$ or it can be finite.

Key question: We know that $\lim_{x\to\infty} f(x) = 0$ but does $f(x)$ go to 0 fast enough? (or $\lim_{x\to-\infty} f(x) = 0$)

e.g. $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$ $\qquad \lim_{x\to\infty} \frac{\ln(x)}{x^2} = 0$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b}\ln b - \frac{1}{b} + 1\right) = 1$$

e.g. $\int_1^\infty \frac{\ln(x)}{x}\,dx$ $\quad \lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

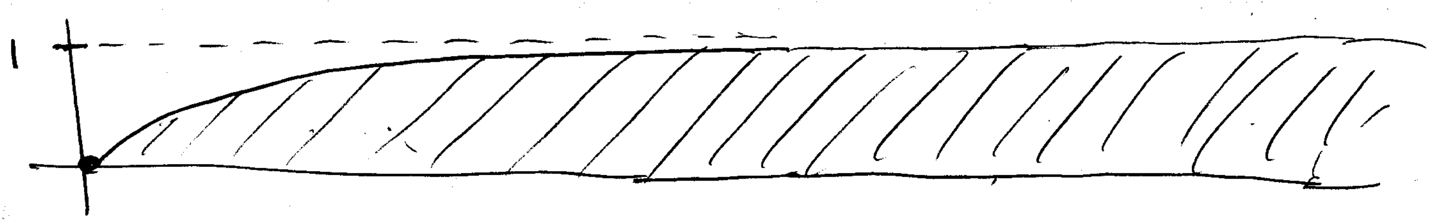
But... $\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$

$= \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 = \infty$

Conclusion: $\frac{\ln(x)}{x^2} \to 0$ fast enough

$\frac{\ln(x)}{x} \to 0$ not fast enough.

e.g. $\int_0^\infty \frac{x}{x+1}\,dx$ $\quad \lim_{x\to\infty} \frac{x}{x+1} = 1$



$\int_0^\infty \frac{x}{x+1}\,dx = \infty$ because $\lim_{x\to\infty} \frac{x}{x+1}$ is not 0.

For $\int_a^\infty f(x)\,dx$ to be finite it is necessary that $\lim_{x\to\infty} f(x) = 0$ but it is <u>not</u> <u>sufficient</u>

Relative rates of growth (at ∞)

1. Exponential growth

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to\infty} 2^x = \infty \qquad \lim_{x\to\infty} \left(\tfrac{1}{2}\right)^x = 0$$

2. Power growth (or polynomial growth)

$$\lim_{x\to\infty} x^3 = \infty \qquad \lim_{x\to\infty} \frac{1}{x^3} = \lim_{x\to\infty} x^{-3} = 0$$

$$\lim_{x\to\infty} x^2+3x+1 = \infty \qquad \lim_{x\to\infty} (x^3+x)^{1/2} = \infty.$$

3. Logarithmic growth

$$\lim_{x\to\infty} \ln(x) = \infty.$$

Basic facts. (a) Exponential growth beats power growth

(b) Power growth beats logarithmic growth.

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \infty, \quad \lim_{x\to\infty} \frac{x^3}{e^x} = 0, \quad \lim_{x\to\infty} \frac{x^{100}}{(1.5)^x} = 0$$

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = 0 \qquad \lim_{x\to\infty} \frac{(\ln x)^{100}}{x^{.001}} = 0$$

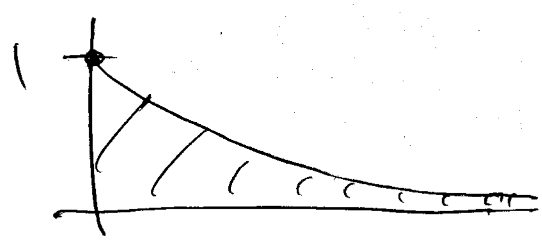
$$\lim_{x\to\infty} \frac{x^{1/2}}{\ln x} = \infty \qquad \lim_{x\to\infty} \frac{2x\cancel{+3}}{\sqrt{x^3\cancel{+2}}} = \lim_{x\to\infty} \frac{2x}{x^{3/2}}$$

$$= \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$$

e.g. $\int_0^\infty e^{-x}\,dx$ $\qquad \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$

$$= \lim_{b\to\infty}\left(\int_0^b e^{-x}\,dx\right) = \lim_{b\to\infty}\left(-e^{-x}\Big|_0^b\right) = \lim_{b\to\infty}(-e^{-b}+1)$$

$$= 1$$



e.g. $\int_0^\infty x^2 e^{-x}\,dx$ $\qquad \lim_{x\to\infty} x^2 e^{-x} = 0$

$$= \lim_{b\to\infty} \int_0^b x^2 e^{-x}\,dx$$

$\int x^2 e^{-x}\,dx$ $\quad u = x^2 \quad dv = e^{-x}dx$, $\quad du = 2x\,dx \quad v = -e^{-x}$

$= -x^2e^{-x} + 2\int x e^{-x}\,dx$ $\quad u = x \quad dv = e^{-x}$, $\quad du = dx \quad v = -e^{-x}$

$= -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right]$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$

$$= \lim_{b \to \infty} \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \Big|_0^b \right)$$

$$= \lim_{b \to \infty} \left(\underbrace{-\cancel{b^2 e^{-b}}}_{0} \underbrace{-\cancel{2b e^{-b}}}_{0} \underbrace{-\cancel{2 e^{-b}}}_{0} + 2 \right)$$

$$= 2$$

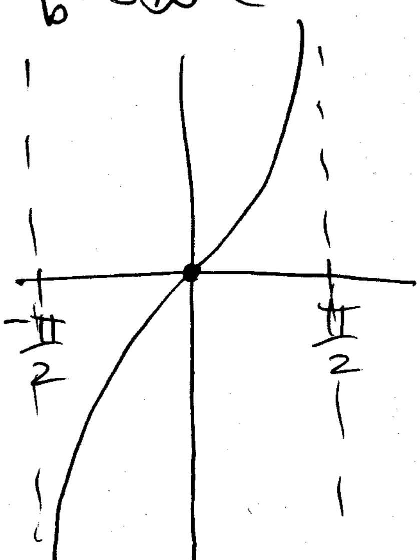
e.g. $\displaystyle\int_0^\infty x^{100} e^{-x}\, dx$ will be finite.

e.g. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\, dx \qquad \lim_{x \to \infty} \frac{1}{x^2+1} = 0$

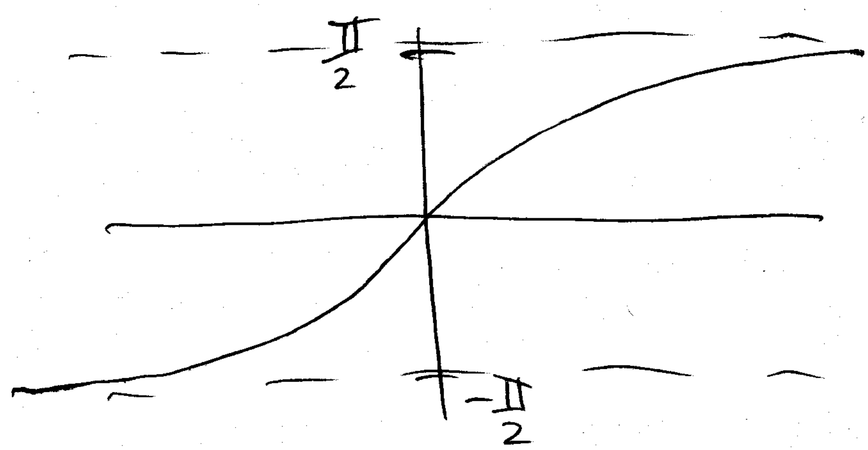
fast enough?

$$= \lim_{b \to \infty} \int_0^b \frac{1}{x^2+1}\, dx = \lim_{b \to \infty} \left(\tan^{-1}(x) \Big|_0^b \right)$$

$$= \lim_{b \to \infty} \left(\tan^{-1}(b) - \underbrace{\cancel{\tan^{-1}(0)}}_{0} \right) = \lim_{b \to \infty} \tan^{-1}(b) = \frac{\pi}{2}$$



$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$



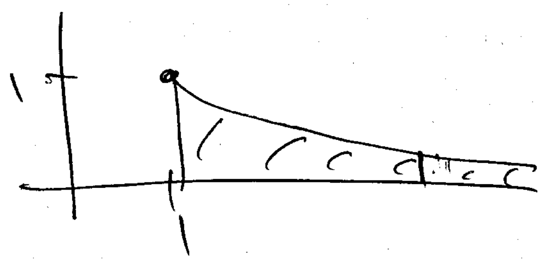
$\tan^{-1}(x)$

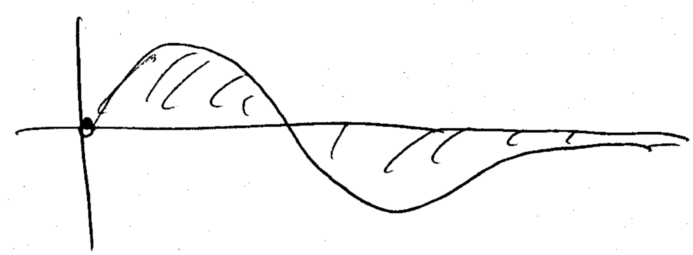
eg $\int_1^\infty \frac{1}{x^{1/2}}\,dx$ $\qquad \lim_{x\to\infty} \frac{1}{x^{1/2}} = 0$ fast enough?

NO

$$= \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx = \lim_{b\to\infty} 2x^{1/2}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(2b^{1/2} - 2\right) = \infty$$





e.g. $\int_1^\infty \frac{1}{x^p}\,dx$

$p > 0$, $p \neq 1$

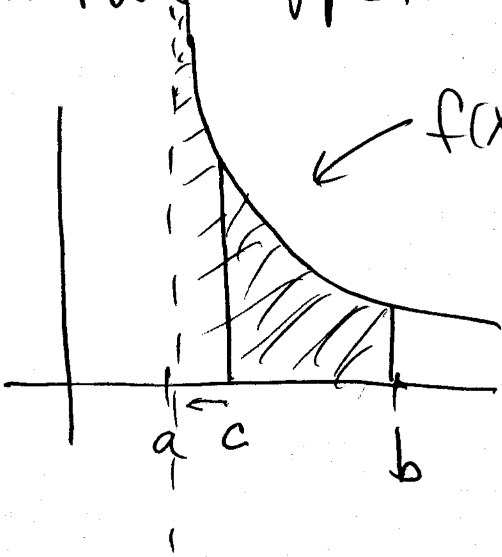
$$= \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \left(\frac{1}{-p+1} x^{-p+1}\right)\Big|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p} b^{1-p} - \frac{1}{1-p}\right) = \begin{cases} \infty & \text{if } 0<p\le 1 \\ \frac{1}{p-1} & \text{if } p>1 \end{cases}$$

$0<p<1 \rightarrow$ not fast enough

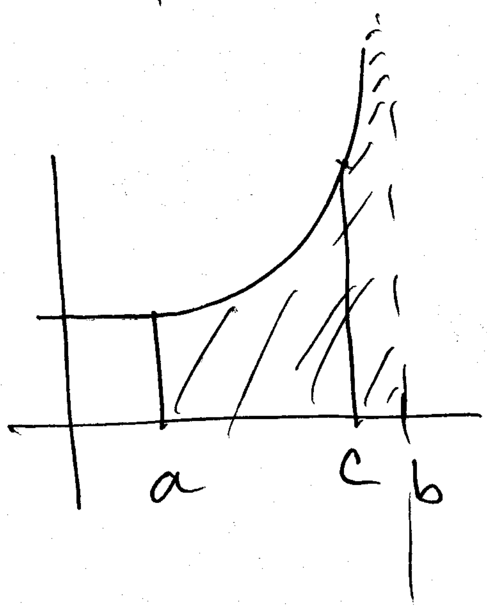
$p>1 \rightarrow$ fast enough.

Another type:



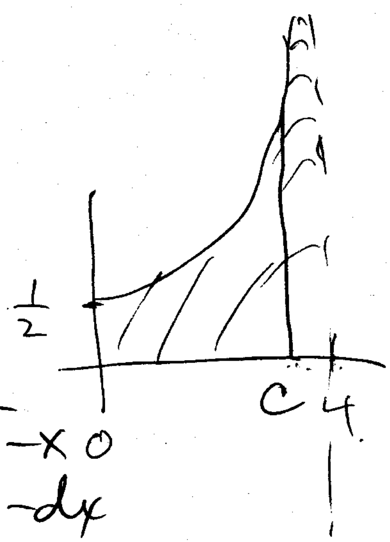
vertical asymptote at $x=a$

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$



$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

e.g. $\displaystyle\int_0^4 \frac{dx}{(4-x)^{1/2}}$ vert. asymp at $x=4$



$$\lim_{c\to 4^-} \int_0^c \frac{dx}{(4-x)^{1/2}}$$

$$= \lim_{c\to 4^-} -2(4-x)^{1/2}\Big|_0^c$$

$-\displaystyle\int \frac{-dx}{(4-x)^{1/2}}$ $\quad u=4-x$, $du=-dx$

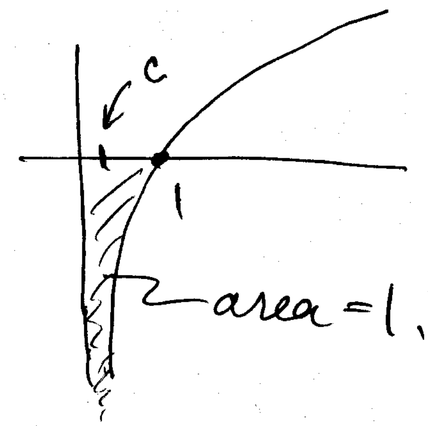
$= -\displaystyle\int \frac{du}{u^{1/2}} = -2u^{1/2}$

$= -2(4-x)^{1/2}$

$$= \lim_{c \to 4^-} -2(\cancel{4-c})^{1/2} + 2(4)^{1/2}$$

$$= 4$$

e.g. $\int_0^1 \ln(x)\,dx$



$$= \lim_{c \to 0^+} \left(\int_c^1 \ln(x)\,dx \right)$$

$$\int \ln(x)\,dx \qquad u = \ln(x) \quad dv = dx$$

$$du = \frac{1}{x}\,dx \quad v = x$$

$$= x\ln(x) - \int dx = x\ln(x) - x$$

$$= \lim_{c \to 0^+} \left(x\ln(x) - x \Big|_c^1 \right) = \lim_{c \to 0^+} 1 \cdot \cancel{\ln(1)} - 1 - c\ln(c) - c$$

$$= \lim_{c \to 0^+} (-1 - c\ln(c) - c) = -1$$

$$\lim_{x \to 0^+} x\ln(x) \qquad 0 \cdot \infty$$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \quad \frac{\infty}{\infty} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -\frac{x^2}{x}$$

$$= \lim_{x \to 0^+} (-x) = 0.$$