Quiz 6 - Section 7.3

Techniques of Integration

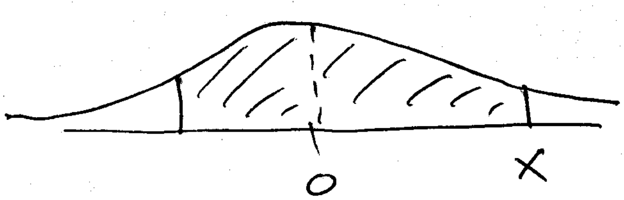
$\int f(x)\,dx$ — How do you find antiderivative of $f(x)$?

Idea: Some anti-derivatives don't exist in "closed form"

e.g. $\int (1+x^4)^{1/2}\,dx$, $\int e^{-x^2}\,dx$, $\int \sin(x^2)\,dx$

do not have closed form expressions.

$e^{-\pi t^2}$

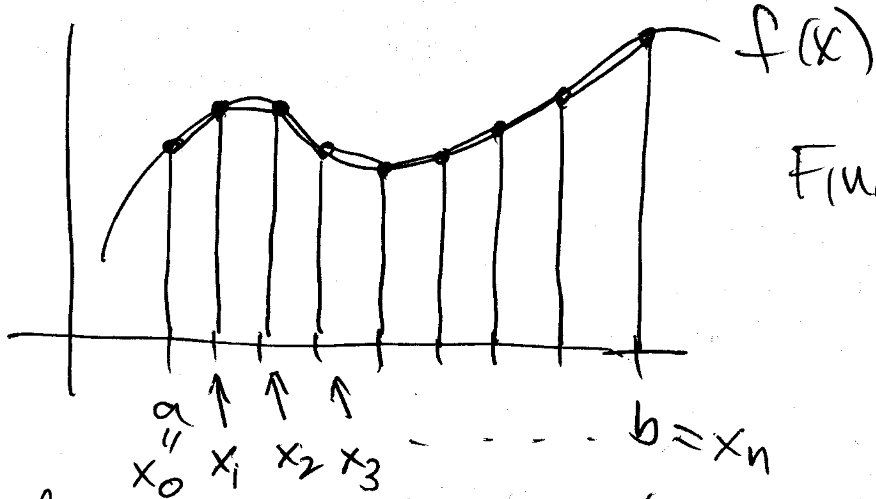


calculating probabilities requires finding integrals like $\int_0^x e^{-\pi t^2}\,dt$ but tables are the only way to do it.

Q! How do they get numbers for the table?

7.6 Numerical Integration

A. Trapezoid Rule.



Find $\int_a^b f(x)\,dx$

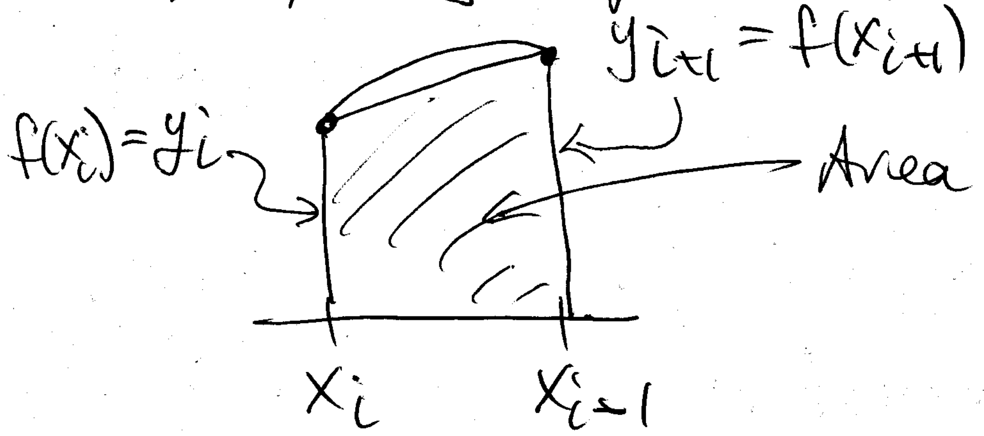
Rule: ① Pick a value for n

② Divide $[a,b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$.

③ Look at values $f(x_i) = y_i$

$i = 0, 1, 2, \ldots, n$

④ Approx area under graph of f over $[x_i, x_{i+1}]$ by a trapezoid.



⑤ Add up all the A_i.

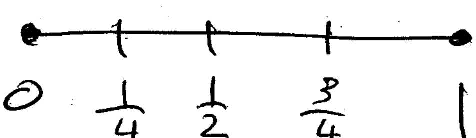
$$A_i = \left(\frac{y_i + y_{i+1}}{2}\right)\underbrace{(x_{i+1} - x_i)}_{\Delta x} = \left(\frac{1}{2}y_i + \frac{1}{2}y_{i+1}\right)\Delta x$$

$$\int_a^b f(x)\,dx \approx \left(\frac{1}{2}y_0 + \frac{1}{2}y_1\right)\Delta x + \left(\frac{1}{2}y_1 + \frac{1}{2}y_2\right)\Delta x$$
$$+ \left(\frac{1}{2}y_2 + \frac{1}{2}y_3\right)\Delta x + \cdots + \left(\frac{1}{2}y_{n-1} + \frac{1}{2}y_n\right)\Delta x$$

$$= \Delta x\left(\frac{1}{2}y_0 + \frac{1}{2}y_1 + \frac{1}{2}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_2 + \frac{1}{2}y_3\right.$$
$$\left. + \cdots + \frac{1}{2}y_{n-1} + \frac{1}{2}y_n\right)$$

$$= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)$$

e.g. $\int_0^1 (t^3 + t)\,dt$ $\quad n = 4$



$\Delta x = \frac{1}{4}$

$f(0) = 0$

$f\left(\frac{1}{4}\right) = \frac{1}{64} + \frac{1}{4} = \frac{17}{64}$

$f\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$

$f\left(\frac{3}{4}\right) = \frac{27}{64} + \frac{3}{4} = \frac{75}{64}$

$f(1) = 2$

$$T(4) = \frac{1}{4}\left(\frac{1}{2}\cdot 0 + \frac{17}{64} + \frac{40}{64} + \frac{75}{64} + \frac{1}{2}\cdot 2\right)$$

$$\begin{array}{r} 75 \\ 57 \\ \hline 132 \end{array}$$

$$= \frac{1}{4}\left(\frac{196}{64}\right) = .765625$$

Exact answer: $\int_0^1 (t^3+t)\,dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 \Big|_0^1$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = .75$$

Absolute error $= \left| T(4) - \int_0^1 t^3 + t\, dt \right|$

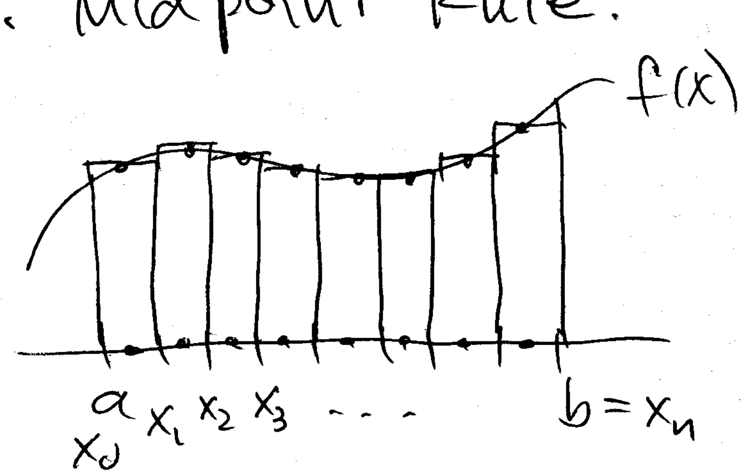
$$= |.765625 - .75| = .015625$$

Pct error $= \dfrac{|\text{Absolute error}|}{|\text{true value}|} = \dfrac{.015625}{.75}$

$$\cong .022$$

2.2% error

2. Midpoint Rule.



$$\Delta x = \frac{b-a}{n}$$

$$\cancel{M(n)\approx\int}\quad\int_a^b f(x)\,dx \approx M(n)$$

$$= f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\Delta x$$

$$= \Delta x\left(\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\right)$$

e.g. $\displaystyle\int_0^1 (t^3+t)\,dt \qquad \Delta x = \frac{1}{4},$

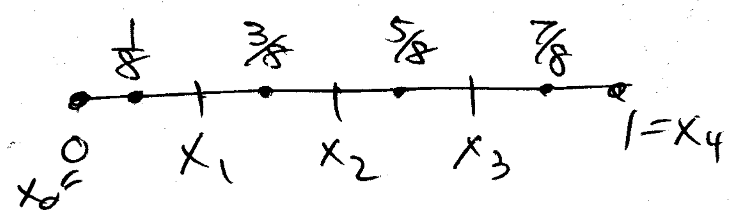
$x_0 = 0$

$x_1 = \frac{1}{4} \to \frac{1}{8}$

$x_2 = \frac{1}{2} \to \frac{3}{8}$

$x_3 = \frac{3}{4} \to \frac{5}{8}$

$x_4 = 1 \to \frac{7}{8}$



$$M(4) = \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$$

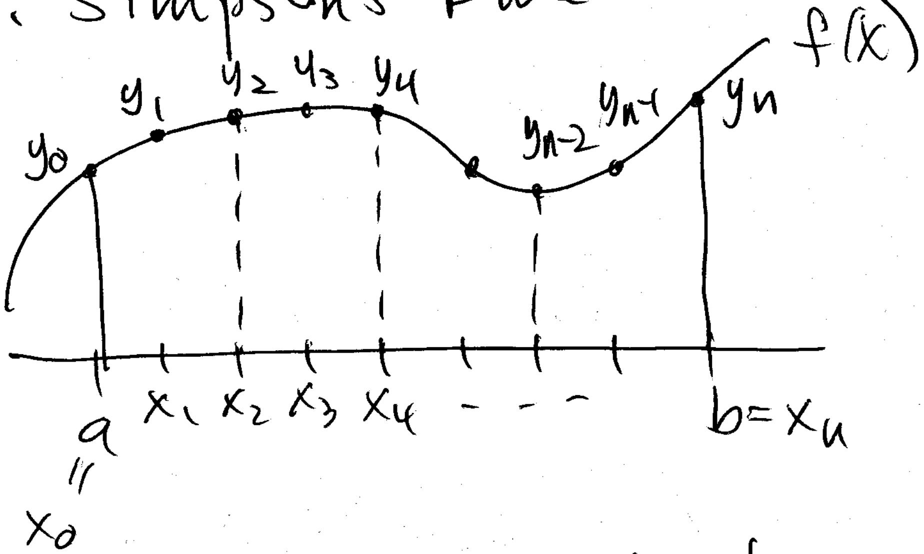
$$= \frac{1}{4}\left(\frac{65}{512} + \left(\frac{27}{512}+\frac{3}{8}\right) + \left(\frac{125}{512}+\frac{5}{8}\right) + \left(\frac{343}{512}+\frac{7}{8}\right)\right)$$

$$= \frac{1}{4}\left(\frac{560}{512} + \frac{15}{8}\right) = \frac{1}{4}\left(\frac{560+960}{512}\right)$$

$$= .7421875$$

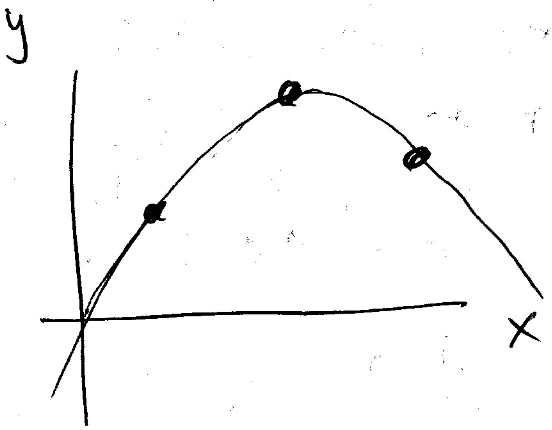
Pct error: $\dfrac{|.7421875 - .75|}{|.75|} \cong .0104$ 1.04% About twice as good as trap.

C. Simpson's Rule.



$$\Delta x = \frac{b-a}{n}$$

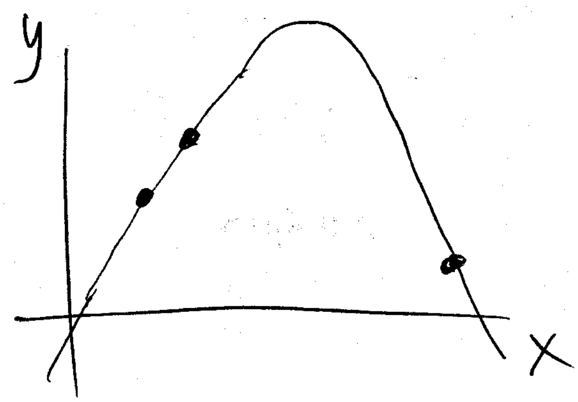
Take points 3 at a time and <u>interpolate</u> a quadratic function between them



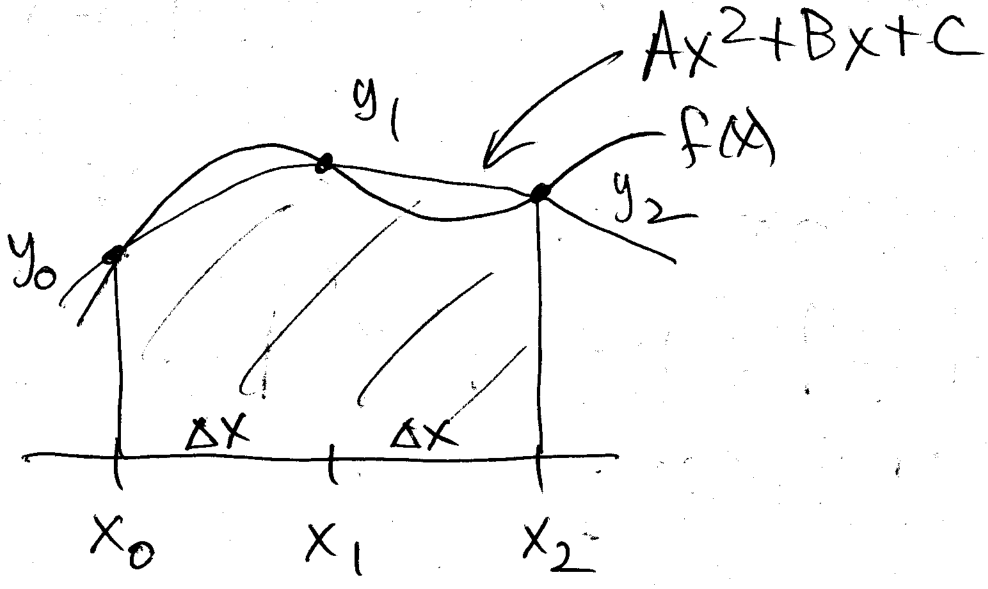
Given 3 points, can I find a quadratic

$$y = Ax^2 + Bx + C$$

that passes through all 3 points? YES.



<u>This problem can be solved.</u>



$$\int_{x_0}^{x_2} (Ax^2+Bx+C)\,dx = \tfrac{1}{3}Ax^3 + \tfrac{1}{2}Bx^2 + Cx \Big|_{x_0}^{x_2}$$

$$= \cdots = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

So adding up gives

$$S(n) = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$

$$+ \frac{\Delta x}{3}(y_4 + 4y_5 + y_6) + \cdots + \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Note: n must be even.

e.g. $\int_0^1 (t^3+t)\,dt \quad n=4$

$\Delta x = \frac{1}{4}$

$x_0 = 0 \qquad y_0 = 0$

$x_1 = \frac{1}{4} \qquad y_1 = \frac{17}{64}$

$x_2 = \frac{1}{2} \qquad y_2 = \frac{5}{8}$

$x_3 = \frac{3}{4} \qquad y_3 = \frac{75}{64}$

$x_4 = 1 \qquad y_4 = 2$

$$S(n) = \frac{1}{12}\left(0 + \underset{4\cdot\frac{17}{64}}{\frac{17}{16}} + \underset{2\cdot\frac{5}{8}}{\frac{5}{4}} + \underset{4\cdot\frac{75}{64}}{\frac{75}{16}} + 2\right) = \frac{1}{\cancel{12}}\left(\frac{\overset{12}{\cancel{144}}}{16}\right)$$

$$= \frac{3}{4}$$

EXACT!