Quiz 6 <u>7.3</u> Trig Substitutions

Integrating rational functions
Partial Fraction Expansions

<u>Idea</u>: Given $\dfrac{P(x)}{Q(x)}$, P, Q polynomials

You can expand this into a sum of simple rational functions that you can integrate.

<u>Key</u>: Factor Q(x).

A. Simple linear factors.

<u>e.g.</u> Expand $\dfrac{x+1}{x^3-x^2-6x}$

$$x^3-x^2-6x = x(x^2-x-6) = x(x-3)(x+2)$$

$$\frac{x+1}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2} \qquad \text{Find } A, B, C$$

$$= \frac{A(x-3)(x+2) + Bx(x+2) + Cx(x-3)}{x(x-3)(x+2)}$$

$$x+1 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

$\underline{x=0}$: $1 = A(-3)(2) = -6A \rightarrow A = \frac{-1}{6}$

$\underline{x=3}$: $4 = B(3)(5) = 15B \rightarrow B = \frac{4}{15}$

$\underline{x=-2}$: $-1 = C(-2)(-5) = 10C \rightarrow C = \frac{-1}{10}$

So to solve;

$$\int \frac{x+1}{x^3-x^2-6x}\,dx = \int -\frac{1}{6}\cdot\frac{1}{x}\,dx + \int \frac{4}{15}\frac{1}{x-3}\,dx$$
$$+ \int \frac{-1}{10}\frac{1}{x+2}\,dx$$
$$= -\frac{1}{6}\int \frac{1}{x}\,dx + \frac{4}{15}\int \frac{1}{x-3}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$
$$= -\frac{1}{6}\ln|x| + \frac{4}{15}\ln|x-3| - \frac{1}{10}\ln|x+2| + C.$$

B. Repeated linear factors.

e.g. $\int \frac{6x+7}{(x+2)^2}\,dx$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

(If we had
$$\frac{6x+7}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}\Big)$$

$$6x+7 = A(x+2)+B$$

$x=-2$: $-5 = B$

Take any x $x=0$! $7 = 2A+B = 2A-5$

$2A = 12$ $A = 6$

OR Take Derivative

$$\frac{d}{dx}(6x+7) = \frac{d}{dx}(\overbrace{A(x+2)+B}^{Ax+2A+B})$$

$6 = A$

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{x+2} - \frac{5}{(x+2)^2}\,dx$$

$$= 6\int \frac{1}{x+2}\,dx - 5\int \frac{1}{(x+2)^2}\,dx$$

$$= 6\ln|x+2| + \frac{5}{(x+2)} + C.$$

$u = x+2$

$du = dx$

$$\int \frac{1}{u^2}\,du = \frac{-1}{u} + c$$

$$= \frac{-1}{(x+2)} + c$$

e.g. $\displaystyle\int \frac{2x^2+x+1}{x(x+1)^2}\,dx$

$$\frac{2x^2+x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$

$$2x^2+x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=0$: $1 = A$

$x=-1$: $2 = -C \rightarrow C = -2$

$\underline{x=1}$: $4 = 4A + 2B + C = 4 + 2B - 2$

$\uparrow$ arbitrary x.

$= 2 + 2B$

$\longrightarrow \underline{B=1}$

$$\int \frac{2x^2+x+1}{x(x+1)^2}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{-2}{(x+1)^2}\,dx$$

$$= \ln|x| + \ln|x+1| + \frac{2}{x+1} + C$$

c. Irreducible quadratic factors.

$\underline{\text{e.g.}}$ $$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t(t^2+1)}\,dt$$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1} = \frac{A(t^2+1)+(Bt+C)(t)}{t(t^2+1)}$$

$$3t^2+t+4 = A(t^2+1) + \underbrace{(Bt+C)(t)}_{Bt^2+Ct}$$

$\underline{t=0}$: $\underline{4 = A}$

Take a derivative:

$$6t+1 = 2At + 2Bt + C$$

$\underline{t=0:}$ $\underline{\underline{1 = C}}$

Do it again:

$$6 = 2A + 2B = 2\cdot 4 + 2B = 8 + 2B$$

$$\longrightarrow \underline{\underline{B = -1}}$$

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t(t^2+1)}\,dt = \int_1^{\sqrt{3}} \frac{4}{t}\,dt + \int_1^{\sqrt{3}} \frac{-t+1}{t^2+1}\,dt$$

$$= 4\ln|t|\Big|_1^{\sqrt{3}} + \frac{-1}{2}\int_1^{\sqrt{3}} \frac{2t}{t^2+1}\,dt + \int_1^{\sqrt{3}} \frac{1}{t^2+1}\,dt$$

$\downarrow$

$u = t^2+1$

$du = 2t\,dt$

$t=1 \quad u=2$

$t=\sqrt{3} \quad u=4$

$$= -\frac{1}{2}\int_2^4 \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u|\Big|_2^4$$

$\downarrow$

$\tan^{-1}(t)\Big|_1^{\sqrt{3}}$

$$= 4\left(\ln(\sqrt{3}) - \cancel{\ln(1)}^{0}\right) - \frac{1}{2}(\ln 4 - \ln 2)$$

$$+ \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= 2\ln(3) - \frac{1}{2}\ln(2) + \frac{\pi}{3} - \frac{\pi}{4}$$

$$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} \quad //$$

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e.g. $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

$$\begin{array}{r|l} & x^2+1+\frac{1}{x^2-1} \\ \hline x^2-1 & x^4+0x^3+0x^2+0x+0 \\ & -x^4 \quad\quad +x^2 \\ \hline & x^2 \\ & -x^2+1 \\ \hline & 1 \end{array}$$

$$= \int\left(x^2+1+\frac{1}{x^2-1}\right)dx$$

$$= \int(x^2+1)\,dx + \int \frac{1}{x^2-1}\,dx$$

$$\hookrightarrow \frac{1}{\underset{(x+1)(x-1)}{x^2-1}} = \frac{A}{x+1} + \frac{B}{x-1}$$

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$$\int \frac{\sqrt{9-x^2}}{x}\,dx = \int \frac{(9-x^2)^{1/2}}{x}\,dx$$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$(9-x^2)^{1/2} = 3\cos\theta$

$$= \int \frac{\cancel{3}\cos\theta}{\cancel{3}\sin\theta}\cdot 3\cos\theta\,d\theta$$

$$= 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

2 ways

$$= 3\int \frac{\cos^2\theta}{\sin^2\theta}\sin\theta\,d\theta$$

$$= 3\int \frac{\cos^2\theta}{1-\cos^2\theta}\,{-\sin\theta}\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= -3\int \frac{u^2}{1-u^2}\,du$$

$$= 3\int \frac{u^2}{u^2-1}\,du$$

$$= 3\int \frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\,du$$

$$= 3\int\left(1 + \frac{1}{u^2-1}\right)du$$

$$= 3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= 3\int \frac{1}{\sin\theta}\,d\theta - 3\int \sin\theta\,d\theta$$

$$= 3\int \csc\theta\,d\theta - 3\int \sin\theta\,d\theta$$

$$\csc\theta = \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta + \cot\theta}$$

$u = \csc\theta + \cot\theta$

$du = -\csc^2\theta - \csc\theta\cot\theta$

$$-3\int \frac{1}{u}\,du \text{ etc....}$$