Quiz 5 - 7.1, 7.2

Integration by parts

Integrals of powers of trig functions

7.3 Trigonometric Substitutions

Used on integrals involving expressions like $(a^2-x^2)^{1/2}$ $(x^2-a^2)^{1/2}$ $(x^2+a^2)^{1/2}$

e.g. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{(4+x^2)^{1/2}}$ Trick: Make a clever substitution

Substitute: $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$(4(1+\tan^2\theta))^{1/2} = 4^{1/2}(1+\tan^2\theta)^{1/2}$

$(4+x^2)^{1/2} = (4+(2\tan\theta)^2)^{1/2} = (4+4\tan^2\theta)^{1/2}$

$= 2(1+\tan^2\theta)^{1/2} = 2(\sec^2\theta)^{1/2} = \cancel{2|\sec\theta|}$

$= 2\sec\theta$

Recall: $\cos^2 x + \sin^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{2}\cancel{\sec\theta}} = \int \sec\theta \, d\theta$$

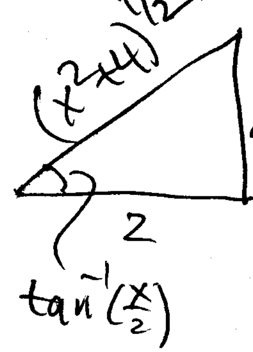
$$= \ln|\sec\theta + \tan\theta| + C$$

Need to substitute back: $x = 2\tan\theta$

$$\theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$= \ln\left|\sec\left(\tan^{-1}\left(\frac{x}{2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x}{2}\right)\right)\right| + C$$

Not simplified enough.



$$\tan\left(\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{x}{2}$$

$$\sec\left(\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{(x^2+4)^{1/2}}{2}$$

$$= \ln\left|\frac{(x^2+4)^{1/2}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left|\frac{x + (x^2+4)^{1/2}}{2}\right| + C$$

---

e.g. $\int (1-9t^2)^{1/2}\, dt = \int \left(9\left(\frac{1}{9} - t^2\right)\right)^{1/2} dt$

$$= 3\int \left(\frac{1}{9} - t^2\right)^{1/2} dt$$

$$t = \frac{1}{3}\sin\theta$$

$$dt = \frac{1}{3}\cos\theta\, d\theta$$

$$\left(\frac{1}{9} - t^2\right)^{1/2} = \left(\frac{1}{9} - \frac{1}{9}\sin^2\theta\right)^{1/2} = \left(\frac{1}{9}(1-\sin^2\theta)\right)^{1/2} = \left(\frac{1}{9}\cos^2\theta\right)^{1/2} = \frac{1}{3}\cos\theta.$$

$$= 3\int \frac{1}{3}\cos\theta \cdot \frac{1}{3}\cos\theta\, d\theta$$

$$= \frac{1}{3}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$$

$$= \frac{1}{3}\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + c$$

$$= \frac{1}{6}\theta + \frac{1}{12}\sin(2\theta) + c$$

$$t = \frac{1}{3}\sin\theta$$

$$3t = \sin\theta$$

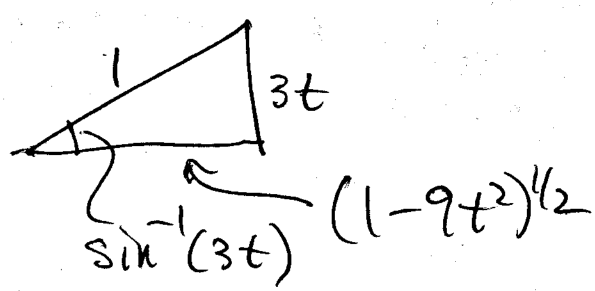
$$\theta = \sin^{-1}(3t)$$

$$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{12}\underbrace{\sin(2\sin^{-1}(3t))} + c$$

Remember $\sin 2x = 2\sin x\cos x$

$$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{12}\cdot 2\sin(\sin^{-1}(3t))\cos(\sin^{-1}(3t)) + c$$

$$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{6}(3t)(1-9t^2)^{1/2} + c$$



$$= \frac{1}{6}\sin^{-1}(\cancel{t}3t) + \frac{1}{2}t(1\cancel{+}9t^2)^{1/2} + c$$

In general:

$(a^2+x^2)^{1/2}$ substitute $x = a\tan\theta$

$(a^2-x^2)^{1/2}$ substitute $x = a\sin\theta$

$(x^2-a^2)^{1/2}$ substitute $x = a\sec\theta$

<u>eg</u> $\displaystyle\int \frac{x^2}{(x^2-1)^{5/2}}\,dx$ $\qquad x = \sec\theta$

$\qquad\qquad\qquad\qquad dx = \sec\theta\tan\theta\, d\theta$

$[(x^2-1)^{1/2}]^5$ $\qquad x^2-1 = \sec^2\theta - 1 = \tan^2\theta$

$\qquad\qquad\qquad\qquad (x^2-1)^{5/2} = (\tan^2\theta)^{5/2} = \tan^5\theta$

$$= \int \frac{\sec^2\theta}{\tan^{4}\theta}\sec\theta\,\cancel{\tan\theta}\,d\theta = \int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$= \int \frac{1}{\cancel{\cos^3\theta}}\cdot\frac{\cos^4\theta}{\sin^4\theta}\,d\theta \qquad \boxed{\sec\theta = \frac{1}{\cos\theta} \quad \tan\theta = \frac{\sin\theta}{\cos\theta}}$$

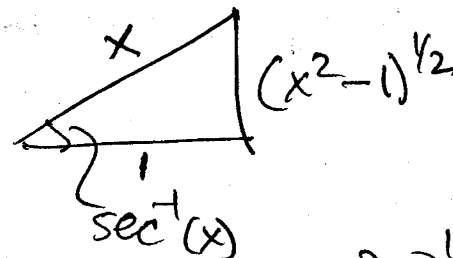
$$= \int \frac{\cos\theta}{\sin^4\theta}\,d\theta \qquad u = \sin\theta, \quad du = \cos\theta\,d\theta$$

$$= \int \frac{1}{u^4}\,du = -\frac{1}{3}\frac{1}{u^3} + c = -\frac{1}{3}\frac{1}{\sin^3\theta} + c.$$

$\int u^{-4}\,du = \frac{1}{-3}u^{-3} + c$

$$x = \sec\theta \longrightarrow \theta = \sec^{-1}(x)$$

$$= -\frac{1}{3}\,\frac{1}{\sin^3(\sec^{-1}(x))} + C$$



$$\sin(\sec^{-1}(x)) = \frac{(x^2-1)^{1/2}}{x}$$

$$= -\frac{1}{3}\left(\frac{x}{(x^2-1)^{1/2}}\right)^3 + c$$

$$= -\frac{1}{3}\,\frac{x^3}{(x^2-1)^{3/2}} + c.$$

<u>e.g</u> 42) $\displaystyle\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2-64}}$

$x = 8\sec\theta$

$dx = 8\sec\theta\tan\theta\,d\theta$

$x^2 - 64 = 64\sec^2\theta - 64$

$= 64(\sec^2\theta - 1)$

$= 64\tan^2\theta$

$(x^2-64)^{1/2} = 8\tan\theta$

$x = 8\sqrt{2} \longrightarrow 8\sqrt{2} = 8\sec\theta$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$x = 16 \longrightarrow 16 = 8\sec\theta$

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{8}\sec\theta\cancel{\tan\theta}}{\cancel{8}\cancel{\tan\theta}}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/3} \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta|\,\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \ln\left(\sec\tfrac{\pi}{3} + \tan\tfrac{\pi}{3}\right) - \ln\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right)$$

$$= \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1)$$

$$= \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right) \;//$$

7.4 Partial Fractions

Idea: How do you integrate rational functions? i.e. $\frac{P(x)}{Q(x)}$, P, Q polynomials.

Key: Need some algebra of polynomials

e.g. $\int \frac{5x-3}{x^2-2x-3}\,dx$

$= \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$

$= 2\ln|x+1| + 3\ln|x-3| + C$

Look!

$x^2-2x-3 = (x+1)(x-3)$

$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)}$

$= \frac{2}{x+1} + \frac{3}{x-3}$

$\frac{2(x-3)+3(x+1)}{(x+1)(x-3)} = \frac{5x-3}{(x+1)(x-3)}$

How did we get the expansion?

Solve: $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Method 1:

$Ax-3A+Bx+B$

$\frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x+(-3A+B)}{(x+1)(x-3)}$

$= \frac{5x-3}{(x+1)(x-3)}$

$A+B=5 \rightarrow B=3$

$-(-3A+B=-3)$

$4A=8 \rightarrow A=2$

Method 2:

As before: $\dfrac{A}{x+1} + \dfrac{B}{x-3} = \dfrac{A(x-3)+B(x+1)}{(x+1)(x-3)}$

$= \dfrac{5x-3}{(x+1)(x-3)}$ we need $A(x-3)+B(x+1) = 5x-3$

$\underline{x=3}$: $4B = 12 \rightarrow B = 3$

$\underline{x=-1}$: $-4A = -8 \rightarrow A = 2$