Exam 1 – Monday 2-18

Coverage 6.1 – 6.8 (omit 6.6)

Calculators will be allowed.

(Only basic scientific features allowed)

One 3x5 card with formulas (both sides)

50 minutes

New material in second half of class.

Oval Reviews

- Thursday + Friday
- Schedule and sign-up info is on web.

Ch 7 – Integration Techniques

7.1 Integration by Parts

Idea: Substitution: Inverting the Chain Rule.

$$\int_a^b f'(g(x))g'(x)\,dx \longrightarrow \int_{g(a)}^{g(b)} f'(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

Parts: Inverting the product rule.

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(u \cdot v)\,dx = \int u \frac{dv}{dx}\,dx + \int v \frac{du}{dx}\,dx$$

shorthand

$$\int d(u \cdot v) = \int u\,dv + \int v\,du$$

$$u \cdot v = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

e.g. $\int x \cos(x)\,dx$

① split integrand into u and dv.

② Want: choose u so that it simplifies when you take its derivative (like a polynomial)

③ Want: choose dv so it is easy to integrate.

$u = x \qquad dv = \cos(x)\,dx$

$du = dx \qquad v = \sin(x)$

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$

$$= x\sin(x) - (-\cos(x)) + C$$

$$= x\sin(x) + \cos(x) + C$$

Check: $\frac{d}{dx}\left(x\sin(x) + \cos(x)\right)$

$$= x\cos(x) + \underbrace{\cancel{\sin(x)} - \cancel{\sin(x)}}\ = x\cos(x).$$

eg $\int \underbrace{x}_{u}\,\underbrace{e^{-x}\,dx}_{dv} = -xe^{-x} - \int -e^{-x}\,dx$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$$= -xe^{-x} - e^{-x} + C$$

e.g. $\int \underbrace{x^2}_{u}\,\underbrace{\sin(x)\,dx}_{dv} = -x^2\cos(x) + \int 2x\cos x\,dx$

$u = x^2 \quad dv = \sin(x)\,dx$

$du = 2x\,dx \quad v = -\cos(x)$

int by parts again

$$= -x^2\cos(x) + 2\left(x\sin(x) + \cos(x)\right) + C.$$

eg

$$\int \underbrace{e^x}_{u} \underbrace{\cos(x)\,dx}_{dv}$$

$$\boxed{\begin{array}{ll} u = e^x & dv = \cos(x)\,dx \\ du = e^x\,dx & v = \sin(x) \end{array}}$$

$$= e^x \sin(x) - \left[\int e^x \sin(x)\,dx\right]$$

$$u = e^x \quad dv = \sin(x)\,dx$$

$$du = e^x\,dx \quad v = -\cos(x)$$

$$= -e^x\cos(x) + \int e^x \cos(x)\,dx$$

$$= e^x \sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$

$$\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$

Solve for $\int e^x \cos(x)\,dx$:

$$2\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x)$$

$$\int e^x\cos(x)\,dx = \frac{1}{2}e^x\left(\sin(x)+\cos(x)\right) + C$$

What if we started with $u = \cos(x)$ $dv = e^x\,dx$?
Would it still work? Yes.

what if we started with ~~$u=\cos(x)$, $dv=e^x dx$~~

$u=e^x$ $dv=\cos(x)dx$ for the first step

(we get $\int e^x\cos(x)dx = e^x\sin(x) - \underline{\int e^x\sin(x)dx}$)

then used $u=\sin(x)$ $dv=e^x dx$ for second step? Would it still work? <u>No</u>

$$\int e^x\sin(x)dx = \underline{e^x\sin(x) - \int e^x\cos(x)dx}$$

$u=\sin(x)$ $dv=e^x dx$

$du=\cos(x)dx$ $v=e^x$

$$\int e^x\cos(x)dx = e^x\sin(x) - \left[e^x\sin(x) - \int e^x\cos(x)dx\right]$$

$$= e^x\cancel{\sin(x)} - e^x\cancel{\sin(x)} + \int e^x\cos(x)dx$$

Nowhere.

e.g.

$$\int \ln(x)\,dx = x\ln(x) - \int x\cdot\frac{1}{x}\,dx$$

$u = \ln(x)$ $dv = dx$

$du = \frac{1}{x}dx$ $v = x$

$$= x\ln(x) - \int dx$$

$$= x\ln(x) - x + C$$

e.g.

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}$$

$u = \sin^{-1}(x)$ $dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $v = x$

substitution
$u = 1-x^2$
etc...

e.g.

$$\frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} 2y\tan^{-1}(y^2)\,dy$$

$u = y^2$
$du = 2y\,dy$
$y = 0 \rightarrow u = 0$
$y = \frac{1}{\sqrt{2}} \rightarrow u = \frac{1}{2}$

$$= \frac{1}{2}\int_0^{1/2} \tan^{-1}(y)\,dy$$

$u = \tan^{-1}(y)$ $dv = dy$

$du = \frac{1}{1+y^2}dy$ $v = y$

$$\rightarrow = \frac{1}{2}\left[y\tan^{-1}(y)\Big|_0^{1/2} - \int_0^{1/2}\frac{y}{1+y^2}\,dy\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) - 0 - \frac{1}{2}\ln\frac{5}{4}\right] = \frac{1}{4}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{4}\ln\frac{5}{4}$$

$$\frac{1}{2}\int_0^{1/2} \frac{2y}{1+y^2}\,dy \qquad u = 1+y^2$$

$$du = 2y\,dy$$

$$y=0 \quad u=1$$

$$y=\frac{1}{2} \quad u=\frac{5}{4}$$

$$= \frac{1}{2}\int_1^{5/4} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_1^{5/4}$$

$$= \frac{1}{2}\ln\frac{5}{4} - \frac{1}{2}\ln 1 = \frac{1}{2}\ln\frac{5}{4}$$

7.2 Trigonometric Integrals

e.g. $\int \sin^3(x)\,dx = \int \sin^2(x)\sin(x)\,dx$

$= -\int (1-\cos^2(x))(-\sin(x)\,dx)$

$\boxed{\sin^2(x)+\cos^2(x)=1}$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$= -\int (1-u^2)\,du = \int (u^2-1)\,du$

$= \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3(x) - \cos(x) + C$

e.g. $\int \sin^5(x)\,dx = \int \sin^4(x)\sin(x)\,dx$

$= \int (1-\cos^2(x))^2 \sin(x)\,dx$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$= -\int (1-u^2)^2\,du$ etc....

e.g. $\int \cos^3(x)\sin^5(x)\,dx$

$$= \int \cos^3(x)\sin^4(x)\sin(x)\,dx$$

$$= \int \cos^3(x)\left(1-\cos^2(x)\right)^2 \sin(x)\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int u^3(1-u^2)^2\,du \text{ etc}\ldots$$

e.g. $\int \cos^3(x)\sin^4(x)\,dx$

$$= \int \cos^2(x)\sin^4(x)\cos(x)\,dx$$

$$= \int (1-\sin^2(x))\sin^4(x)\cos(x)\,dx$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$= \int (1-u^2)u^4\,du \text{ etc}\ldots$$

e.g.

$$\int \cos^2(x) \sin^4(x)\, dx$$

$$= \int (1-\sin^2(x)) \sin^4(x)\, dx$$

ok but no help.

Trick:

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx$$