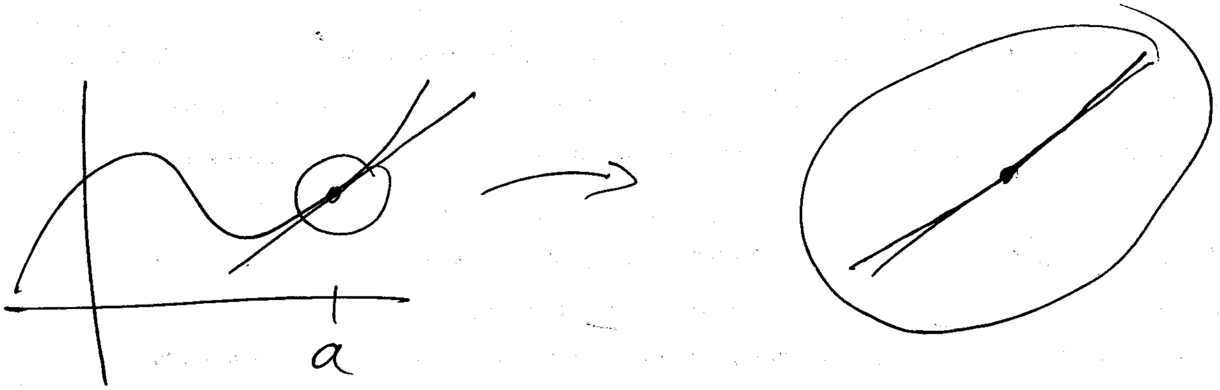
Quiz 9 - 4.3, 4.4

Main idea - the graph of a function follows the tangent line at a point a for all x near a.



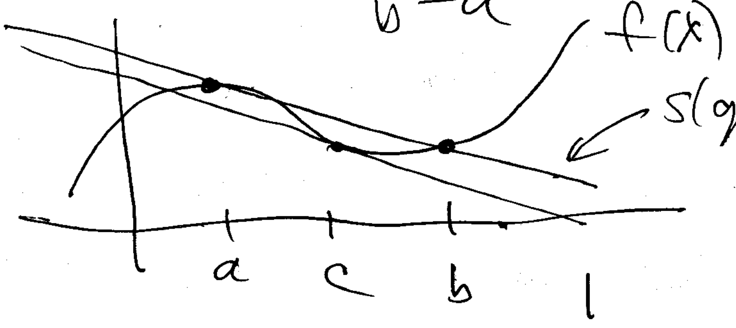
① Linearization

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$, <u>x near a</u>

② Mean value Theorem

- relates <u>avg rate of change</u> and <u>instantaneous rate of change</u>: How?
- average rate of change between $x=a$ and $x=b$ is equal to instantaneous rate of change at some point c in between a and b.

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$ for some c in (a,b).



slope = avg r.o.c. = $\frac{f(b)-f(a)}{b-a}$

4.7 L'Hopital's Rule

Indeterminate form

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

form of limit is $\frac{0}{0}$
value of limit could be anything (or not exist)

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to} (x+1) = 2$$

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3}$$

form: $\frac{0}{0}$
indeterminate

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x+3)} = \lim_{x\to 1} \frac{1}{x+3} = \frac{1}{4}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

form: $\frac{0}{0}$.

L'Hopital's Rule: If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$. In this case

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

e.g. $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{2x}{1} = \lim_{x\to 1} 2x = 2$

e.g. $\lim_{x\to 1}\frac{x-1}{x^2+2x-3} = \lim_{x\to 1}\frac{1}{2x+2} = \frac{1}{4}$

e.g. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

Why this works: suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. So lets say

$f(a) = g(a) = 0$

$$L_f(x) = \underset{0}{\cancel{f(a)}} + f'(a)(x-a) = f'(a)(x-a)$$

$$L_g(x) = \underset{0}{\cancel{g(a)}} + g'(a)(x-a) = g'(a)(x-a)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \left(1+\frac{x}{2}\right)}{x^2} \qquad \text{form } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} \qquad \text{form } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = \frac{-\frac{1}{4}\cancelto{1}{(1)^{-3/2}}}{2} = -\frac{1}{8} \,//$$

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other indeterminate forms:

$\boxed{\frac{\infty}{\infty}}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0

e.g. $$\lim_{x \to \infty} \underbrace{\frac{x^2-1}{x^2+2x+3}}_{\text{form: } \frac{\infty}{\infty}} = \lim_{x \to \infty} \frac{x^2}{x^2} = 1$$

e.g. $$\lim_{x \to \infty} \underbrace{\frac{x-1}{x^2+2x+3}}_{\text{form: } \frac{\infty}{\infty}} = \lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0$$

L'Hopital's Rule: If $\lim_{x \to a} f(x) = \pm\infty$

and $\lim_{x \to a} g(x) = \pm\infty$ (here a can be ∞)

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$$\underbrace{\lim_{x \to \infty} \frac{x^2-1}{x^2+2x+3}}_{\text{form } \frac{\infty}{\infty}} = \underbrace{\lim_{x \to \infty} \frac{2x}{2x+2}}_{\text{form } \frac{\infty}{\infty}} = \lim_{x \to \infty} \frac{2}{2} = 1$$

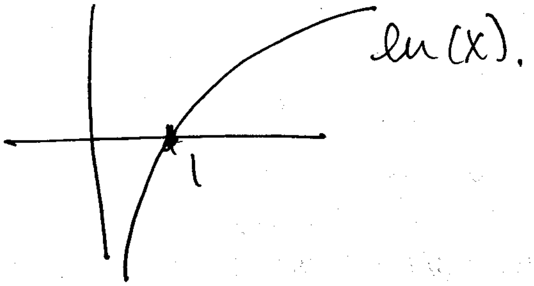
$$\underbrace{\lim_{x \to \infty} \frac{x-1}{x^2+2x+3}}_{\text{form } \frac{\infty}{\infty}} = \lim_{x \to \infty} \frac{1}{2x+2} = 0$$

Other indeterminate forms

$0 \cdot \infty$

$\lim_{x \to 0^+} x \ln(x)$ form $0 \cdot \infty$

$\lim_{x \to 0^+} x = 0$ $\lim_{x \to 0^+} \ln(x) = -\infty$



Why is this indeterminate?
Idea: competing functions.

$f(x) = x$ $g(x) = \ln(x)$ $x \to 0^+$

looking at $x \cdot \ln(x)$

$f(x) = x$ is trying to make $x \cdot \ln(x)$ go to 0

$g(x) = \ln(x)$ is trying to make $x \cdot \ln(x)$ go to $-\infty$

Q: is $x \to 0$ faster than $\ln(x) \to -\infty$? or slower?

e.g. $$\lim_{x\to 1} \frac{x^2-1 \nearrow 0}{x-1 \to 0} = 2$$ $x^2-1 \to 0$ as fast as $x-1 \to 0$ (as $x \to 1$)

$$\lim_{x\to\infty} \frac{x^2-1 \nearrow \infty}{x-1 \to \infty} = \infty$$

So $x^2-1 \to \infty$ faster than $x-1 \to \infty$. (as $x \to \infty$)

To solve we convert it to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$\frac{0}{0}$: $$\lim_{x\to 0^+} \frac{x}{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} \frac{x}{(\ln(x))^{-1}}$$

$$= \lim_{x\to 0^+} \frac{1}{-(\ln(x))^{-2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} \frac{-x}{(\ln(x))^{-2}} \quad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{-1}{-2(\ln x)^{-3}\cdot\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{2(\ln x)^{3}} \quad \frac{0}{0}$$

This will never work.

$\frac{\infty}{\infty}$: $\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$

$= \lim_{x\to 0^+} \frac{1}{x} \cdot -x^2 = \lim_{x\to 0^+} -x = 0$

$\lim_{x\to 0^+} x \ln(x) = 0$

"x goes ~~to~~ to zero faster than ln(x) goes to ∞."

0^0 . $\lim_{x\to 0^+} x^x$ ~~How~~ How do you do this?

$\ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} \ln(x^x)$

$= \lim_{x\to 0^+} x \ln(x) = 0$

$\therefore \lim_{x\to 0^+} x^x = e^0 = 1$ //

$$\lim_{x \to 1^+} x^{\left(\frac{1}{x-1}\right)} \quad \text{form } 1^\infty$$

$$\ln\left(\lim_{x \to 1^+} x^{\frac{1}{x-1}}\right) = \lim_{x \to 1^+} \ln\left(x^{\frac{1}{x-1}}\right)$$

$$= \lim_{x \to 1^+} \frac{1}{x-1} \ln(x) = \lim_{x \to 1^+} \frac{\ln x}{x-1} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = \lim_{x \to 1^+} \frac{1}{x} = 1$$

$$\lim_{x \to 1^+} x^{\left(\frac{1}{x-1}\right)} = e^1 = e$$