Sections 4.1–4.4

Main point: $f'(x)$ tells us where $f(x)$ is increasing ($f'>0$) and decreasing ($f'<0$). If $f'(a)=0$ then $x=a$ is a critical point, i.e. the location of a possible local max or min.

Sections 4.5–4.7

Main point: $f(x)$ is approximated by the tangent line of f at $x=a$,
$L(x)=f(a)+f'(a)(x-a)$, for x near a.
$L(x)$ is the linearization of f at $x=a$.

Differential

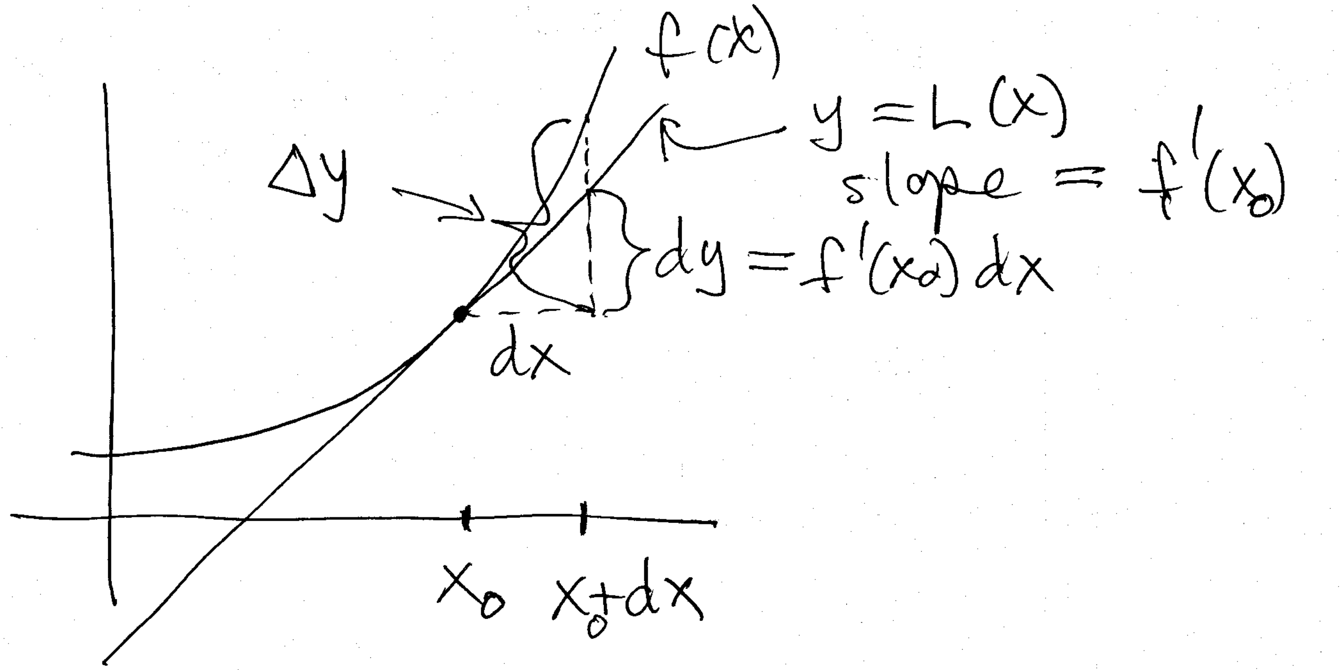
The differential dy of a function $y=f(x)$ is $dy=f'(x).dx$.

① We already have notation $\frac{dy}{dx}=f'(x)$

$\frac{dy}{dx}$: A small change in x of dx induces a small change in y of dy.

② Precisely:

dy = change in $L(x)$ when x goes from x to $x + dx$.



③ The exact change in $f(x)$ is

$\Delta y = f(x_0 + dx) - f(x_0)$. Then

$dy \approx \Delta y$. if dx is small.

#16) Approximate $\sqrt[3]{65}$

Idea: $\sqrt[3]{65} \approx \sqrt[3]{64} = 4$

What is the difference?

$f(x) = x^{1/3}$ $a = 64 = x_0$ $dy \approx$ the difference

$dx = 1$

$\underline{f'(x) = \frac{1}{3}x^{-2/3}}$ $\quad$ $f'(x_0) = f'(64) =$

$\frac{1}{3}(64)^{-2/3} = \frac{1}{3}\cdot\frac{1}{16}$

$= \frac{1}{48}$

$dy = f'(64)\,dx = \frac{1}{48}(1) = \frac{1}{48}$

$dy \approx \Delta y \longrightarrow \Delta y \approx \frac{1}{48} \approx .0208$

So $\sqrt[3]{65} \approx 4.\underbrace{0208}_{dy \approx \Delta y.}$

#14) $\tan(3^\circ) \approx \tan(0^\circ) = 0$

How close to 0?

$y = \tan(x)$ $\quad$ $dy = \sec^2(x)\,dx$

$\boxed{\tan(3^\circ) \approx .052}$ $\quad$ when $x = 0$:

$dy = \sec^2(0)\,dx$

$= dx$

$\Delta y \approx dy = dx$ $\quad$ $dx = 3^\circ = \frac{3}{180}\pi = \frac{\pi}{60}$

$\Delta y \approx \frac{\pi}{60} \approx .052$
$\quad\quad 3$

4.6 Mean Value Theorem.

Ideas: ① Different way of seeing how a function resembles its tangent line.

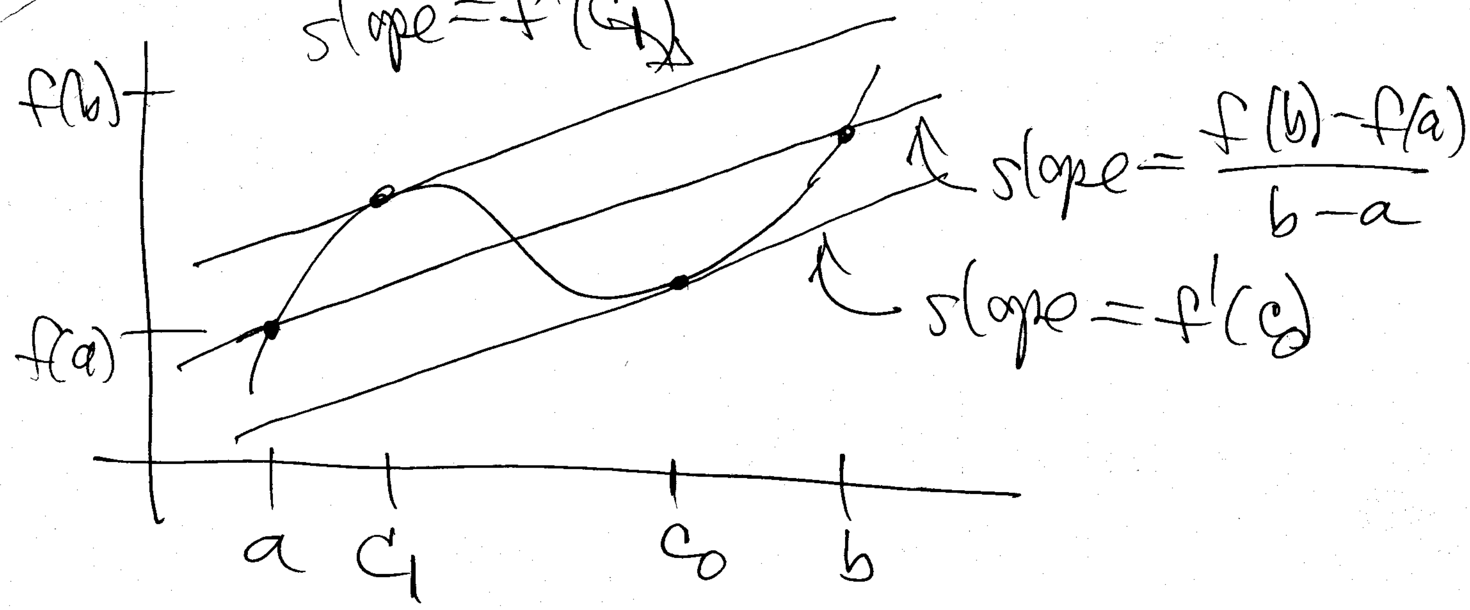
② Relates average rate of change to instantaneous rate of change.

③ The only functions whose derivatives are zero everywhere are constant.

MVT: If f is continuous on [a,b] and differentiable on (a,b) then there is a point c in (a,b) such that

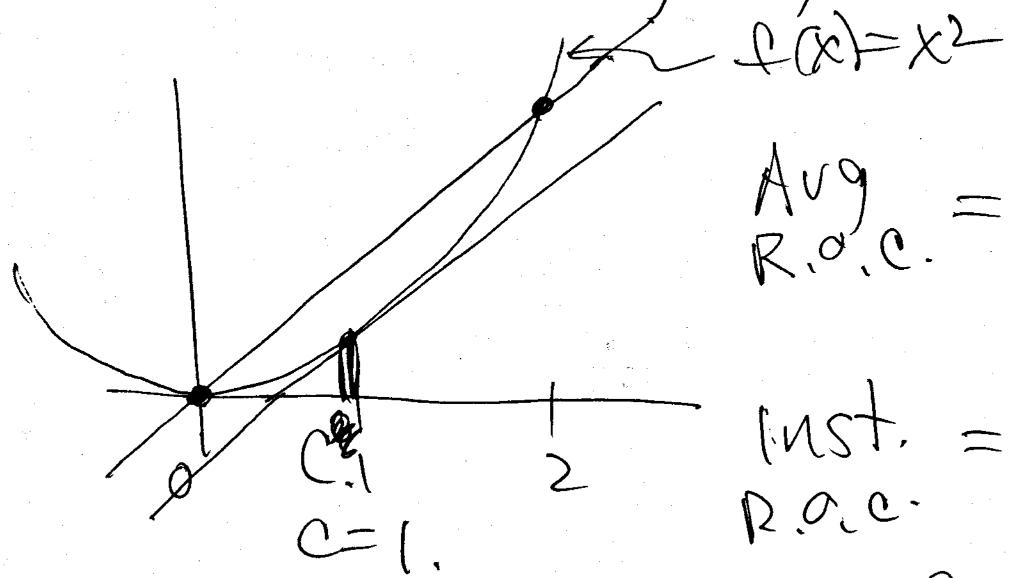
$$\underbrace{\frac{f(b)-f(a)}{b-a}}_{\substack{\text{average rate of}\\ \text{change of f on}\\ [a,b]}} = \underbrace{f'(c)}$$

← inst. rate of change at c in (a,b) (c is some unknown point.)



e.g. $f(x) = x^2$ $[0, 2]$

Find c satisfying MVT.



Avg R.o.c. $= \dfrac{f(2)-f(0)}{2-0} = \dfrac{4-0}{2}$

$= 2$

Inst. R.o.c. $= f'(x) = 2x$

solve: $2c = 2$

$c = 1.$

e.g. $f(x) = \ln(x-1)$ $\quad [2,4]$.

$$\text{Avg R.O.C.} = \frac{f(4)-f(2)}{4-2} = \frac{\ln(3) - \cancel{\ln(1)}^{\,0}}{2}$$

$$= \frac{1}{2}\ln(3)$$

Inst. R.O.C: $f'(x) = \dfrac{1}{x-1}$

Solve: $\dfrac{1}{x-1} = \dfrac{1}{2}\ln(3)$

$$\frac{2}{x-1} = \ln 3$$

$$\frac{2}{\ln 3} = x-1 \longrightarrow x = 1 + \frac{2}{\ln 3} = c.$$

Is c in $(2,4)$?

$1 + \dfrac{2}{\ln 3} \approx 2.82$ so YES!