Quiz 8, 4.1, 4.2

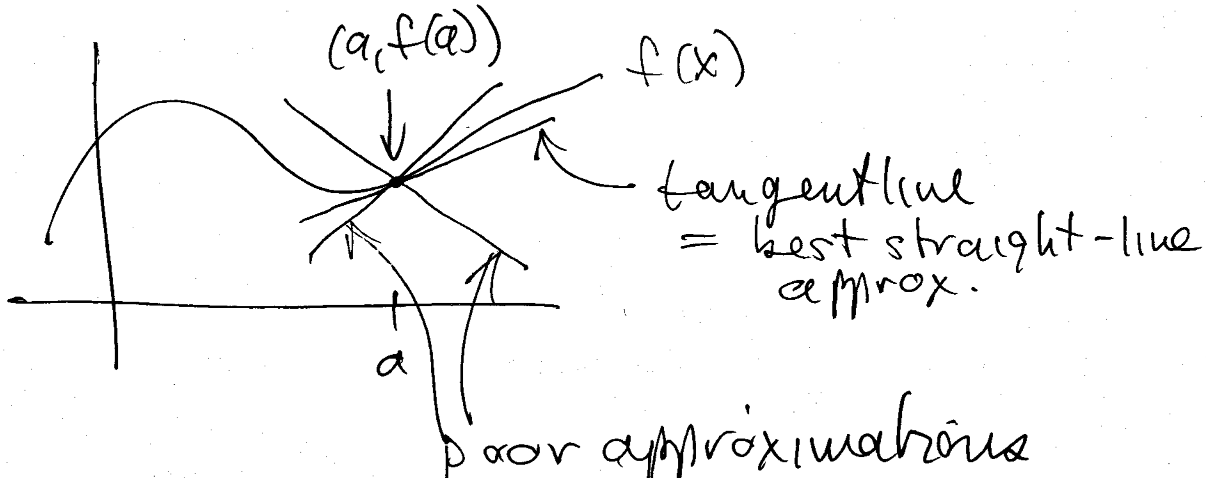
4.5 Linear Approximations (& Differentials)

Idea: Interpretations of $f'(a)$:

① instantaneous rate of change at $x=a$

② slope of the tangent line at $(a, f(a))$

③ the tangent line is the best straight line approximation to $f(x)$ near $x=a$. NEW!



Another way to think of this:

- graph $f(x)$ and tangent line on same axes
- zoom in to point $(a, f(a))$
- two curves become indistinguishable

Linearization of $f(x)$:

Given $f(x)$ and $x=a$

Tangent line: slope $\to f'(a)$

point $\to (a, f(a))$

$$y - f(a) = f'(a)(x-a)$$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

Def: The linearization of $f(x)$ at $x=a$ is $\boxed{L(x) = f(a) + f'(a)(x-a)}$

e.g. Find linearization of $f(x) = (1+x)^{1/2}$ at $x=0$. $f(0)=1$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}(1) = \frac{1}{2}(1+x)^{-1/2}$$

$$f'(0) = \frac{1}{2} \quad \therefore L(x) = 1 + \frac{1}{2}(x-0) = \frac{1}{2}x + 1 \quad //$$

e.g. $f(x) = (x^2+9)^{1/2}$ at $x=4$

$$f(4) = (16+9)^{1/2} = 5$$

$$f'(x) = \frac{1}{\cancel{2}}(x^2+9)^{-1/2}(\cancel{2}x) = x(x^2+9)^{-1/2}$$

$$f'(4) = 4(16+9)^{-1/2} = \frac{4}{5} \therefore L(x) = 5 + \frac{4}{5}(x-4)$$

$$= \frac{4}{5}x + \frac{9}{5} \quad //$$

Approximating functions

$f(x) = (1+x)^{1/2}$, $L(x) = \frac{1}{2}x + 1$

$f(x) \approx L(x)$ if x is near 0.

e.g. $f(.2) = \sqrt{1.2} \approx 1.095445115\ldots$

$L(.2) = \frac{1}{2}(.2) + 1 = 1.1$ close.

$f(-.2) = \sqrt{.8} \approx .894427191$

$L(-.2) = \frac{1}{2}(-.2) + 1 = .9$ close.

$f(2) = \sqrt{3} \approx 1.732050808$

$L(2) = \frac{1}{2}(2) + 1 = 2$ not really close.

How close is close? Percentage error.

e.g: $f(.2)$ close to $L(.2)$?

Absolute error $|f(.2) - L(.2)| \cong .004554$

$$\text{Pct error} = \left|\frac{\text{Absolute error}}{\text{True value}}\right| (\times 100)$$

$$\left|\frac{f(.2) - L(.2)}{f(.2)}\right| = \frac{.004554}{1.095445115} \approx .0042$$

Pct error $\approx .42\%$.

$f(-.2)$ close to $L(-.2)$?

$$\left|\frac{f(-.2)-L(-.2)}{f(-.2)}\right| = \left|\frac{.894427191-.9}{.894427191}\right| \approx .0062$$

Pct error $\approx .62\%$.

$f(2)$ close to $L(2)$?

$$\left|\frac{f(2)-L(2)}{f(2)}\right| = \left|\frac{1.732050808-2}{1.732050808}\right|$$

$$\approx .155$$

Pct error $\approx 15.5\%$

eg #16) $\sqrt[3]{65}$ $\qquad f(x) = x^{1/3}$

find $L(x)$ for $x=a$ ← pick a good a

$$L(x) = f(a) + f'(a)(\underset{65}{x}-a)$$

a must be ① near 65 ② $f(a)$, $f'(a)$ easy to compute.

$a=64$ is good here.