Quiz 6 – 3.4, 3.5

Exam 2 – Oct. 31

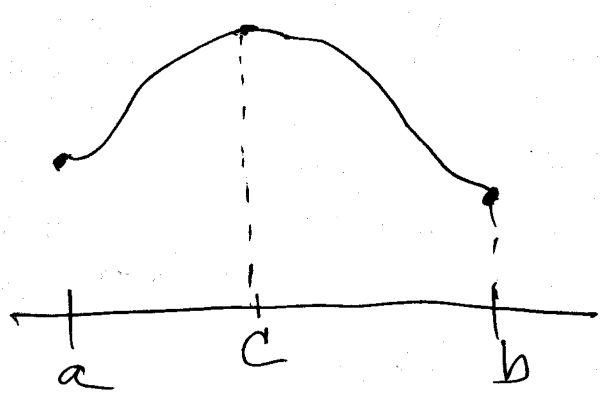
Interpretations of $f'(x)$:

① slope of tangent line to graph of f at x.

② instantaneous rate of change of f with respect to x.

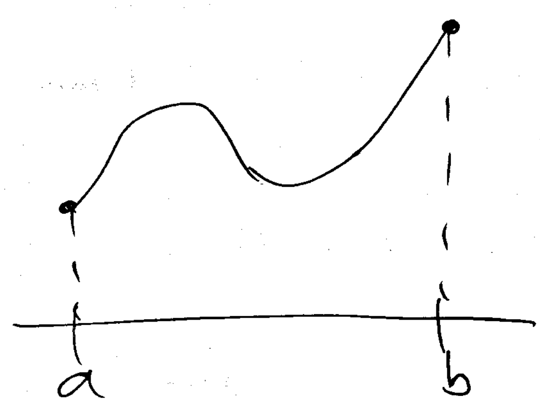
4.1 Maxima and Minima.

Def: A function $f(x)$ on an interval I has an absolute maximum at $x=c$ if $f(x) \leq f(c)$ for all x in I



$I=[a,b]$

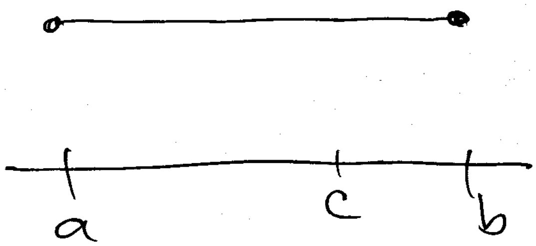
Absolute maximum at $x=c$. Absolute minimum at $x=b$.



$I=[a,b]$

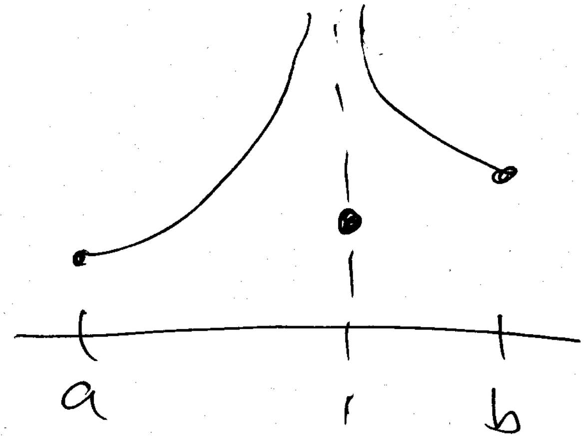
Abs. max at $x=b$.

Abs. min at $x=a$



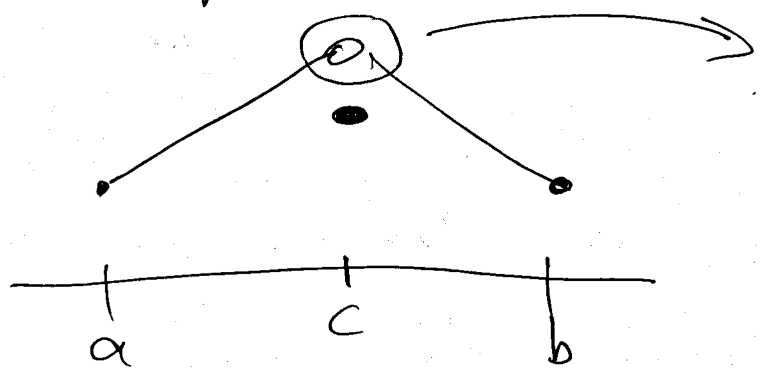
$I=[a,b]$

Abs. max at all points c in $[a,b]$. Abs. min at all points also.



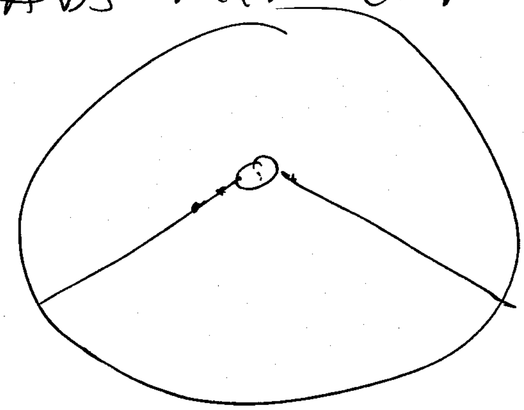
$I=[a,b]$

No abs max on I. Abs min at $x=a$.



$I=[a,b]$

No abs. max on I.



For any number c in I, I can always make f bigger than $f(c)$.

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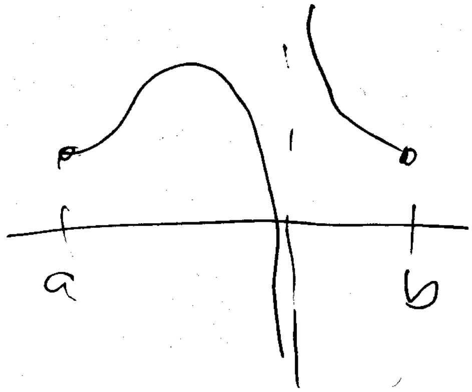
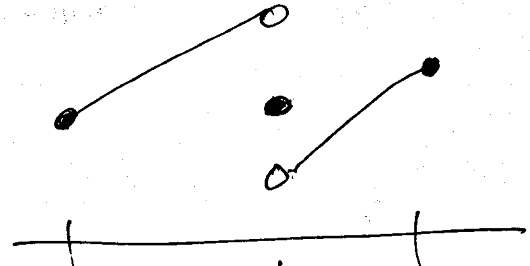
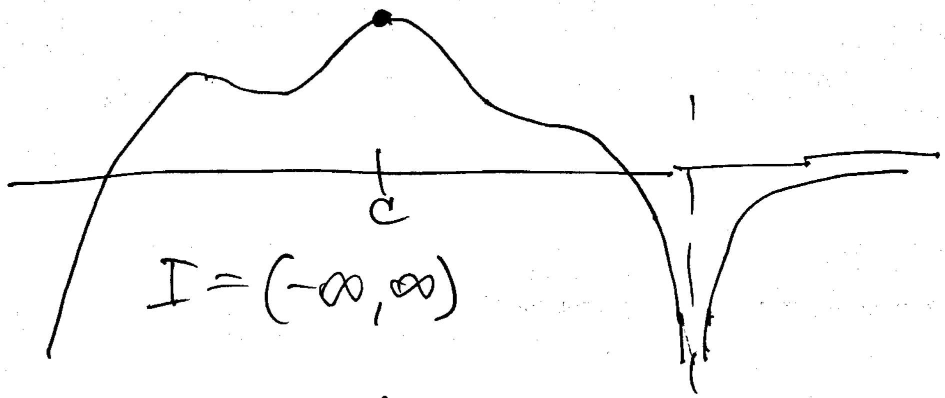
f has an absolute minimum at $x=c$ if $f(x) \geq f(c)$ for all $x \in I$.



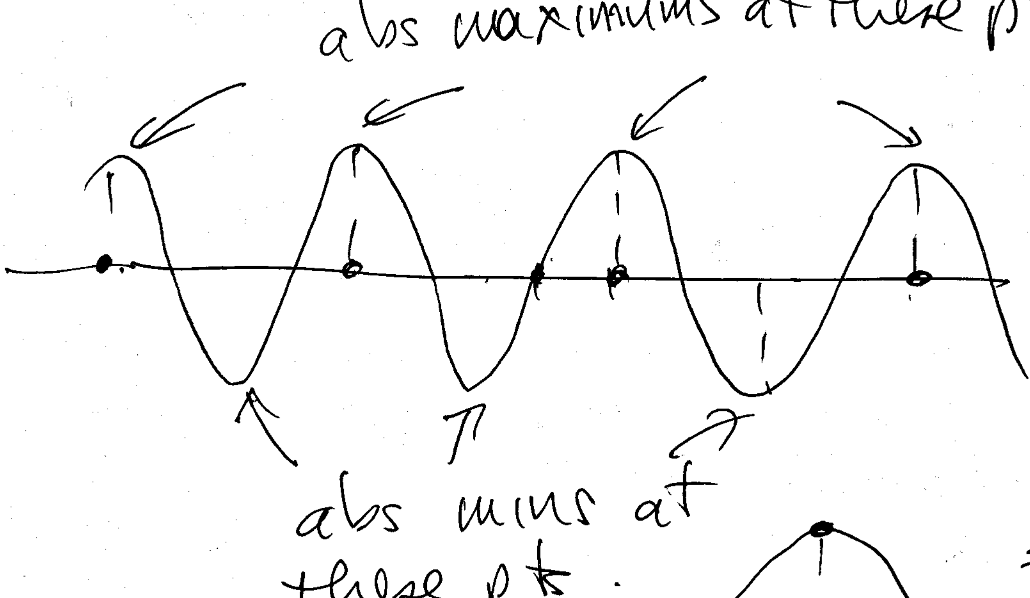
No abs max or min.



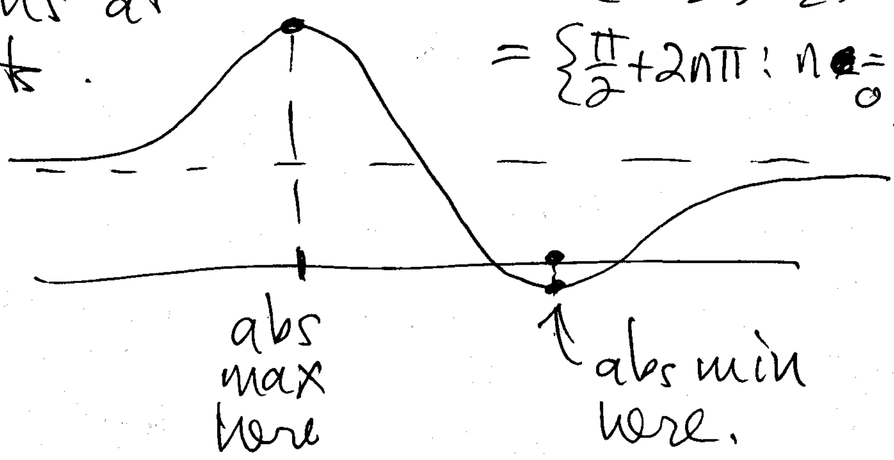
No abs max or min.

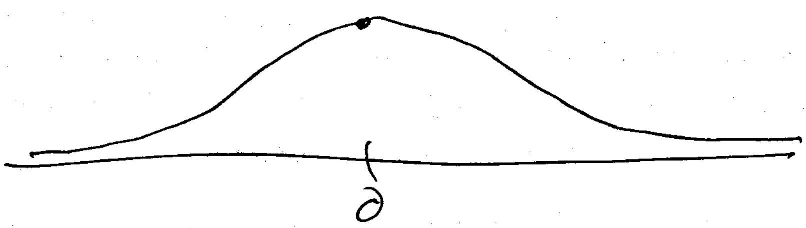


Abs max at $x = c$.
No abs min on I.



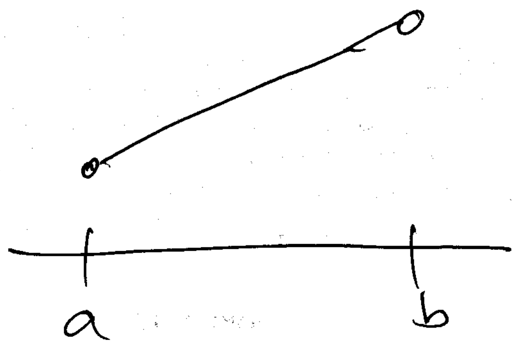
$\sin(x)$
Abs max at
$\{\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots\}$
$\cup \{-\frac{3\pi}{2}, -\frac{7\pi}{2}, -\frac{11\pi}{2}, \dots\}$
$= \{\frac{\pi}{2} + 2n\pi : n = 0, \pm 1, \pm 2, \dots\}$





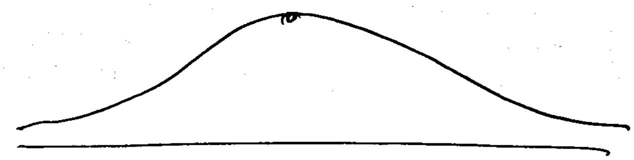
Abs max at $x = 0$.
No abs min on $(-\infty, \infty)$.

Thm: If f is continuous on $I=[a,b]$ then f has an absolute max and min.



$I=(a,b)$

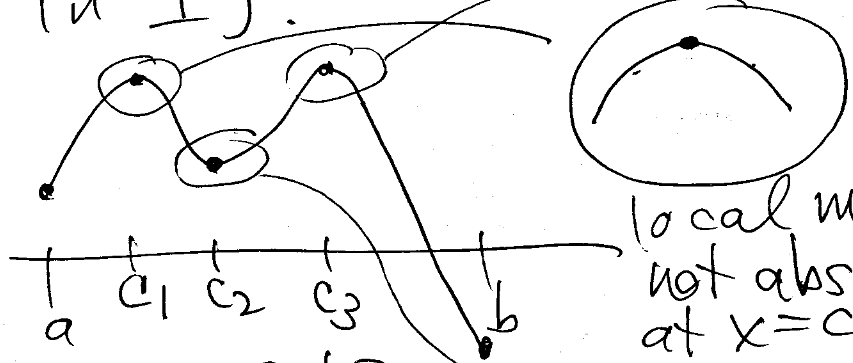
No max or min.



$I=(-\infty,\infty)$

f is cont but no abs minimum.

Def: A function $f(x)$ on interval I has a local maximum/minimum at $x=c$ if $f(x)\le f(c)$ / $f(x)\ge f(c)$ for all x near c (not necessarily for all x in I).



local max and abs max at $x=c_3$

local max not abs. max at $x=c_1$

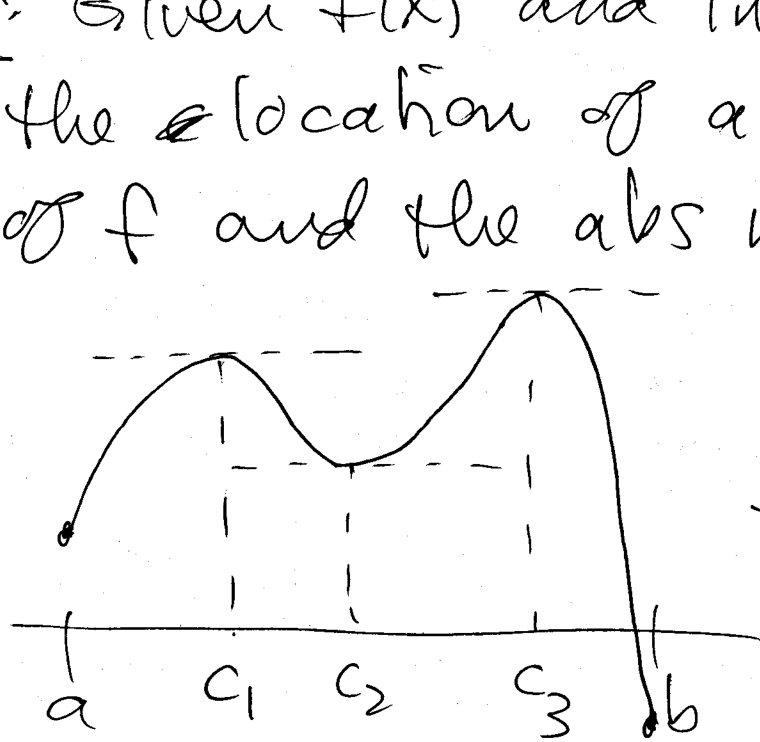
local min not abs min. at $x=c_2$

$I=[a,b]$.

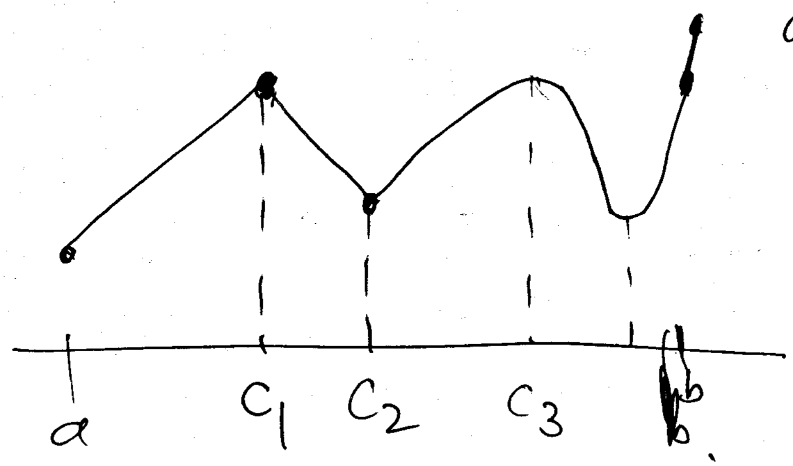
Abs min at b.

Goal: Given $f(x)$ and interval I, find the ~~e~~ location of all local max, min of f and the abs max + min of f.

Idea:



Note: $f'(c_1)=0$, $f'(c_2)=0$, $f'(c_3)=0$, i.e. tangent line is horizontal here. And abs min is at an end point.



Note: $f'(c_1)$ and $f'(c_2)$ do not exist but still are local max and min.

Conclusion: The local maxima and minima and the ~~loca~~ absolute maxima and minima occur either at:

1. end points of interval or
2. points where $f'=0$ or
3. points where f' does not exist

Def: A critical point of f is a value c where $f'(c)=0$ or $f'(c)$ does not exist

Fact: All local max ~~and~~ or min of f occur at critical points.

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e.g. #21) f' undefined at $x=1$ and $x=3$

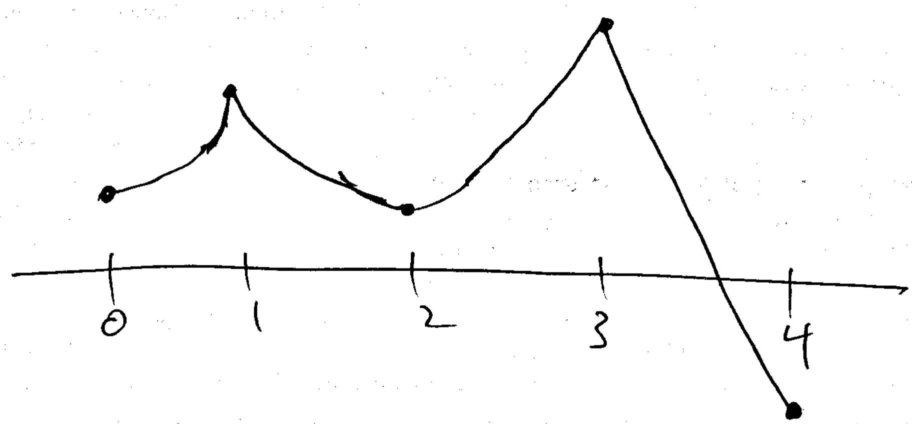
$f'(2)=0$

local max at $x=1$

local min at $x=2$

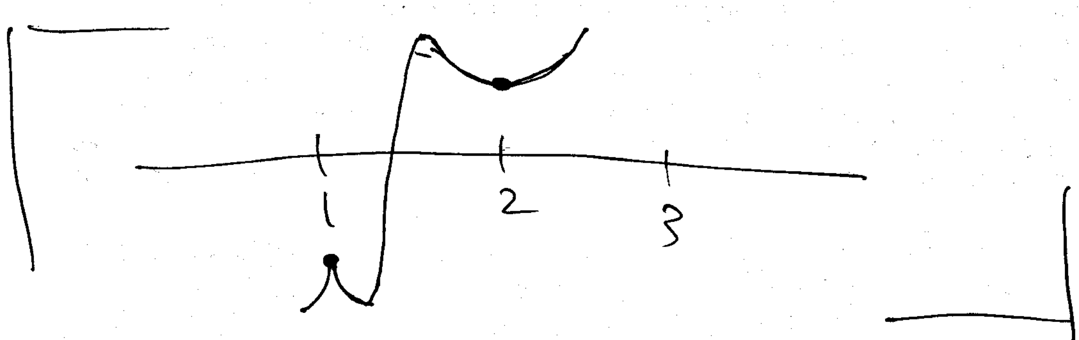
abs max at $x=3$

abs min at $x=4$

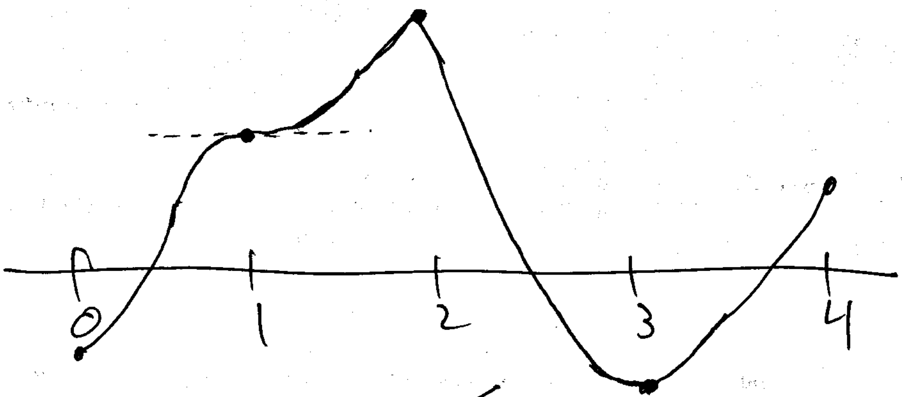


OK

lots of other solutions



#22)



$f'(x) = 0$ at $\underline{x=1}$, $x=3$ ✓

$f'(2)$ undefined ✓

abs max at $x=2$ ✓

→ neither local min nor max at $x=1$ ✓

abs min at $x=3$ ✓

$\underline{eg}$ #26) $f(x) = 12x^5 - 20x^3$ on $[-.5, 2]$

Find all critical pts:

$f'(x) = 60x^4 - 60x^2$

Solve $60x^4 - 60x^2 = 0$

$x^4 - x^2 = 0$

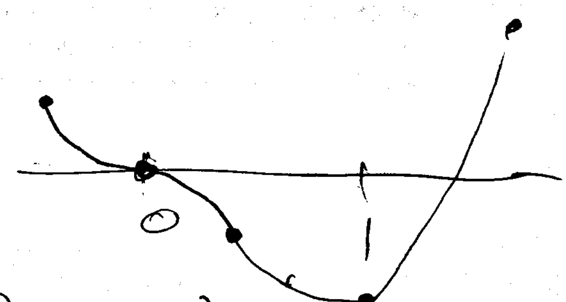
$x^2(x^2-1) = 0$

$x^2(x-1)(x+1) = 0$

Crit pts: $\cancel{x=-1}$, $x=1$, $x=0$

neither local max nor min.

Q: Are $x=1$, $x=0$ local max or min?



$f(0) = 0$

$f(\frac{1}{2}) = \frac{12}{32} - \frac{20}{8} < 0$

$f(-\frac{1}{2}) = \frac{-12}{32} + \frac{20}{8} > 0$

$f(1) = 12 - 20 = -8.$

$$f\left(\tfrac{3}{2}\right) = \cancel{12}^{\,3} \cdot \frac{3^5}{\cancel{32}_{\,8}} - 20 \cdot \frac{27}{8} = \frac{3^6 - 20 \cdot 3^3}{8} = \frac{27}{8}(3^3 - 20)$$

$$= \frac{27}{8}(27-20)$$

$$= \frac{27 \cdot 7}{8} > 0.$$

#30) $f(x) = x - \tan^{-1}(x)$

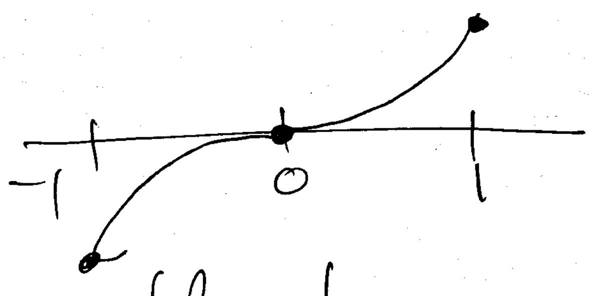
Crit pts: $f'(x) = 1 - \dfrac{1}{1+x^2}$

$$1 - \frac{1}{1+x^2} = 0 \rightarrow 1 = \frac{1}{1+x^2}$$ ← crit pt.

$$1 + x^2 = 1 \rightarrow x^2 = 0 \rightarrow \underline{\underline{x = 0}}$$

Q: Is $x=0$ local max, min, <u>neither</u>?

$$f(0) = 0 - \tan^{-1}(0) = 0 - 0 = 0$$



$$f(1) = 1 - \tan^{-1}(1) = 1 - \frac{\pi}{4} > 0$$

neither loc max nor min.

$$f(-1) = -1 - \tan^{-1}(-1) = -1 - \left(-\frac{\pi}{4}\right) = -1 + \frac{\pi}{4} < 0$$

#34) $f(x) = \dfrac{x}{(x^2+1)^2}$ on $[-2,2]$

Find absolute max/min values of f.

① Find critical pts.

② Check f at crit pts & end points.

$$f'(x) = \frac{(x^2+1)^2(1) - (x)(2(x^2+1)(2x))}{(x^2+1)^4}$$

$$= \frac{\cancel{(x^2+1)}[(x^2+1)-4x^2]}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-3x^2+1}{(x^2+1)^3}$$

Solve: $-3x^2+1=0$

$3x^2=1$

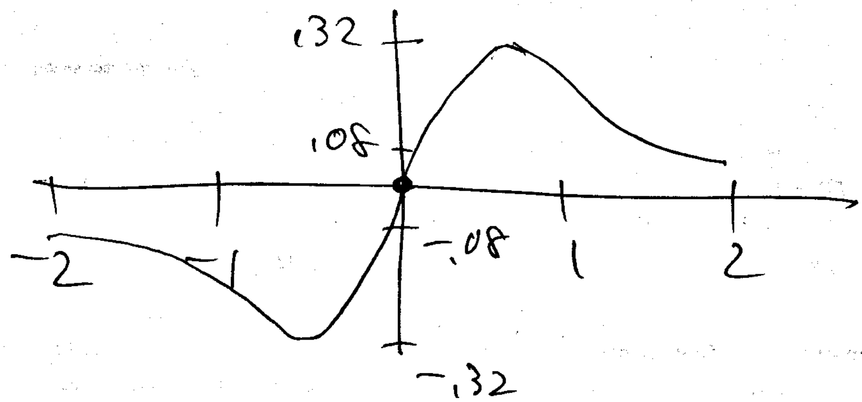
$x = \pm\sqrt{\frac{1}{3}}$ ← critical points

$f(\sqrt{\tfrac{1}{3}}) = \dfrac{\sqrt{\frac{1}{3}}}{(4/3)^2} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{9}{16} = \dfrac{9}{16\sqrt{3}} \approx .32$ ← abs max

$f(-\sqrt{\tfrac{1}{3}}) = -9/16\sqrt{3} \approx -.32$ ← abs min.

$f(2) = \dfrac{2}{25} = .08$

$f(-2) = \dfrac{-2}{25} = -.08$


.32
.08
-.08
-.32
-2
-1
1
2