Quiz 6 – 3.4, 3.5

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \rightarrow e^{\ln(x)} = x$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \rightarrow \sin(\sin^{-1}(x)) = x$$

In general: Given $f(x)$ with inverse $f^{-1}(x)$. Then $\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Another way to look at it:

If $\underline{(x_0, y_0)}$ is on graph of f then $f(x_0) = y_0$ so $f^{-1}(y_0) = x_0$

So $(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$.

e.g. #33 $f(x) = \tan(x)$ $\qquad (1, \pi/4)$

Want:

$\cancel{\tan(1) = \frac{\pi}{4}}$

$\tan(\frac{\pi}{4}) = 1$ ✓

$$(f^{-1})'(1) = \frac{1}{f'(\frac{\pi}{4})}$$

x $\qquad$ $f^{-1}(x)$

$(1, \frac{\pi}{4})$ is on graph of f^{-1}.

$f^{-1}(1) = \frac{\pi}{4}$

$$= \frac{1}{\sec^2(\frac{\pi}{4})}$$

$$= \cos^2(\frac{\pi}{4}) = \frac{1}{2}$$

Another way:

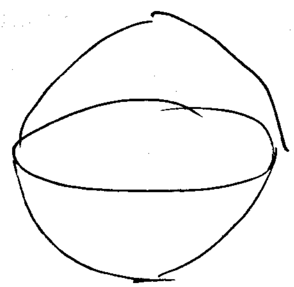
$f^{-1}(x) = \tan^{-1}(x)$ $\qquad (f^{-1})'(x) = \frac{1}{1+x^2}$

$$(f^{-1})'(1) = \frac{1}{1+(1)^2} = \frac{1}{2}$$

3.10 Related Rates.

Idea: If two quantities are related then their rates of change are related.

e.g.



sphere (balloon)

S = surface area of sphere

r = radius of sphere.

Say r is a function of time (t).

$\frac{dr}{dt}$ = rate of change of radius wrt time in e.g. $\frac{cm}{sec}$.

Since S is related to r,

$\frac{dS}{dt}$ is related to $\frac{dr}{dt}$. How?

$$\frac{dS}{dt} \quad \uparrow \quad \frac{cm^2}{sec} \qquad\qquad \frac{dr}{dt} \quad \uparrow \quad \frac{cm}{sec}$$

In fact $S = 4\pi r^2$

So $$\frac{dS}{dt} = \frac{d}{dt}\left(4\pi r^2\right) = \underbrace{8\pi r}_{\frac{dS}{dr}} \frac{dr}{dt}$$

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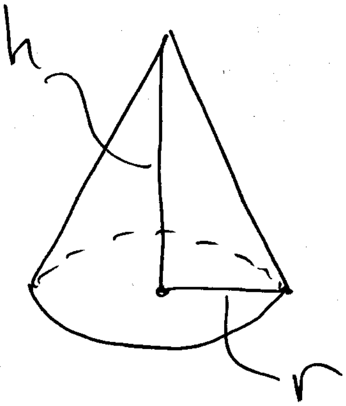
e.g.



V = volume of cone (cm^3)
h = height of cone (cm)
r = radius of cone (cm)

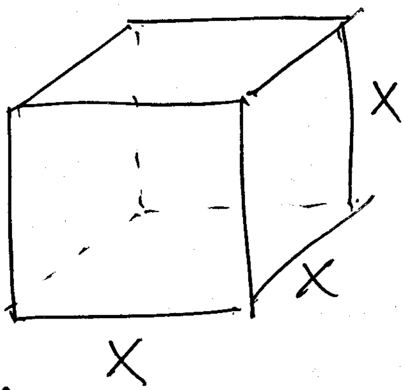
V is related to h and r, so

$\frac{dV}{dt}$ is related to $\frac{dh}{dt}$ and $\frac{dr}{dt}$. How?

In fact: $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + h\cdot 2r\frac{dr}{dt}\right)$$
$$= \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

e.g. #6)



V = volume of cube (cm^3)

x = side length (cm)

Want $\frac{dV}{dt}$ when $x = 50$ and $\frac{dx}{dt} = 2$

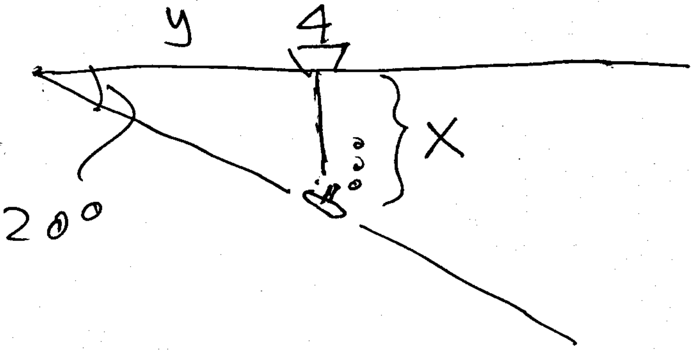
$$V = x^3$$

$$\frac{dV}{dt} = \frac{d}{dt}(x^3) = 3x^2 \frac{dx}{dt}$$

$$\left.\frac{dV}{dt}\right|_{x=50} = 3(50)^2(2)$$

$$= 15000 \ \frac{cm^3}{sec}$$

eg #15)



x = depth of submarine (km)
y = position of boat (km)

Want $\frac{dx}{dt}$ when $\frac{dy}{dt} = 10$

$$\tan(20°) = \frac{x}{y} \rightarrow x = y \cdot \tan(20°)$$

$$\frac{d}{dt}(x) = \frac{d}{dt}(y \tan 20°)$$

$$\frac{dx}{dt} = \frac{dy}{dt} \cdot \tan(20°)$$

$$\frac{dx}{dt} = 10 \tan(20°)$$