Rules for Differentiation

- Product Rule
- Quotient Rule
- Chain Rule ← very important!
- Polynomials
- $\frac{d}{dx} e^x = e^x$
- Trig functions
- Natural logarithm $\frac{d}{dx} \ln(x) = \frac{1}{x}$

How did we get $\frac{d}{dx} \ln(x) = \frac{1}{x}$?

Used: $e^{\ln(x)} = x$

+ implicit diff.

One more formula: $\boxed{\frac{d}{dx}(x^r) = r x^{r-1} \text{ for any number } r}$

$$y = x^r = e^{\ln(x^r)} = e^{r\ln(x)}$$

$$\frac{d}{dx}(x^r) = \frac{d}{dx}\left(e^{r\ln(x)}\right) = e^{r\ln(x)} \cdot \frac{r}{x} = r x^r \cdot \frac{1}{x}$$

$$= r x^{r-1}$$

$$\boxed{e^{g(x)} \xrightarrow{d/dx} e^{g(x)} \cdot g'(x)}$$

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$y = x^{-\frac{5}{7}}$ $\quad \frac{dy}{dx} = -\frac{5}{7} x^{-\frac{5}{7}-1} = -\frac{5}{7} x^{-\frac{12}{7}}$

$y = x^{\sqrt{2}}$ $\quad \frac{dy}{dx} = \sqrt{2}\, x^{(\sqrt{2}-1)}$

Logarithmic Diff.

e.g. $y = \left[(x^2+1)(x-2)^{3/2}\right]^{1/2}$ ← We can do it but its complicated

Idea:

$$\ln(y) = \ln\left(\left[(x^2+1)(x-2)^{3/2}\right]^{1/2}\right)$$

$$= \frac{1}{2} \ln\left[(x^2+1)(x-2)^{3/2}\right]$$

$$= \frac{1}{2}\left[\ln(x^2+1) + \ln(x-2)^{3/2}\right]$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\cdot\frac{3}{2}\ln(x-2)$$

$$= \frac{1}{2}\ln(x^2+1) + \frac{3}{4}\ln(x-2)$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(\frac{1}{2}\ln(x^2+1) + \frac{3}{4}\ln(x-2)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cancel{2}}\frac{1}{x^2+1}\cdot\cancel{2}x + \frac{3}{4}\frac{1}{x-2}$$

$$\frac{dy}{dx} = y \cdot \left(\frac{x}{x^2+1} + \frac{3}{4} \frac{1}{x-2} \right)$$

$$= \left[(x^2+1)(x-2)^{3/2} \right]^{1/2} \left(\frac{x}{x^2+1} + \frac{3}{4} \cdot \frac{1}{x-2} \right)$$

e.g. $f(x) = \dfrac{x^8 \cos^3 x}{(x-1)^{1/2}}$. Find $f'(x)$.

$$\ln f(x) = \ln \left(\frac{x^8 \cos^3 x}{(x-1)^{1/2}} \right)$$

$$= \ln (x^8) + \ln (\cos^3 x) - \ln (x-1)^{1/2}$$

$$= 8 \ln x + 3 \ln (\cos x) - \frac{1}{2} \ln (x-1)$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{8}{x} + \frac{3}{\cos x} \cdot (-\sin x) - \frac{1}{2} \frac{1}{x-1}$$

$$= \frac{8}{x} - 3 \tan (x) - \frac{1}{2(x-1)}$$

$$f'(x) = \frac{x^8 \cos^3 x}{(x-1)^{1/2}} \left(\frac{8}{x} - 3 \tan (x) - \frac{1}{2(x-1)} \right)$$

e.g.

$$y = x^x$$

$$\ln y = \ln(x^x) = x \ln(x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) \cdot 1$$

$$= 1 + \ln(x)$$

$$\frac{dy}{dx} = y \cdot (1 + \ln(x)) = x^x (1 + \ln(x)).$$

$$\boxed{\frac{d}{dx} \ln(y) = \frac{1}{y} \cdot \frac{dy}{dx} \leftarrow \text{chain rule}}$$

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln(a)}$$

$a > 0$.

Recall:

$$\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$\ln(a)$

What is this number?

$$a^x = e^{\ln(a^x)} = e^{x \ln(a)}$$

$$\frac{d}{dx}(a^x) = e^{x\ln(a)} \cdot \frac{d}{dx}(x \ln(a))$$

$$= e^{x \ln(a)} \cdot \ln(a) = \underline{\underline{a^x \ln(a)}}$$

$$\boxed{\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}}$$

$a > 0$

Recall: $a^y = x \iff y = \log_a(x)$

So: $a^{\log_a(x)} = x$

$$\frac{d}{dx}\left(a^{\log_a(x)}\right) = \frac{d}{dx}(x)$$

$$a^{\log_a(x)} \cdot \ln(a) \cdot \frac{d}{dx}(\log_a(x)) = 1$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{\ln(a)\cdot \underbrace{a^{\log_a(x)}}_{x}} = \frac{1}{x\ln(a)}$$

e.g. $y = \dfrac{40}{1+2^{-t}}$ $\qquad \frac{d}{du}2^{u} = 2^{u}\ln(2)$

$$\frac{dy}{dt} = \frac{\cancel{(1+2^{-t})(0)} - 40\left(2^{-t}\ln(2)\cdot(-1)\right)}{(1+2^{-t})^2}$$

$$= \frac{40\ln(2)\cdot 2^{-t}}{(1+2^{-t})^2}$$

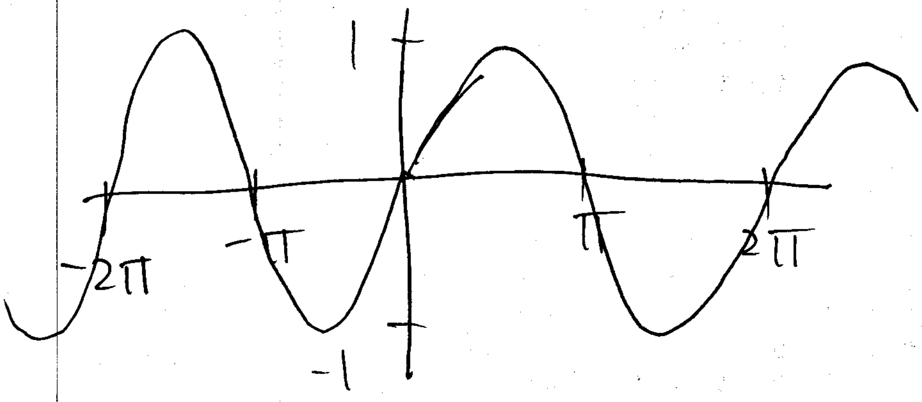
OR $y = 40(1+2^{-t})^{-1}$

$$\frac{dy}{dt} = 40\left(-(1+2^{-t})^{-2}(2^{-t}\ln(2)(-1))\right)$$

$$= 40\ln(2)\,2^{-t}(1+2^{-t})^{-2}.$$

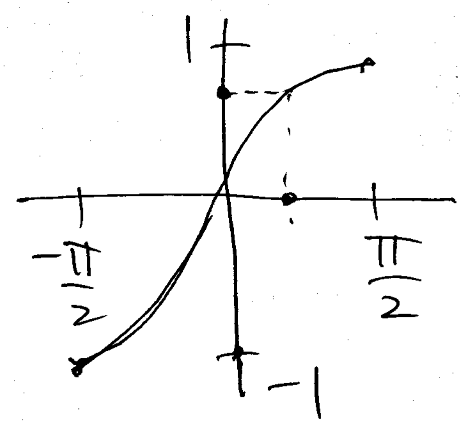
3.8 Inverse Trig Functions

$\sin^{-1}(x) = \arcsin(x)$



not one-to-one

So restrict the domain:



$y = \sin^{-1}(x) \iff x = \sin(y)$

$-1 \le x \le 1$ $\qquad$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $\qquad$ this is one-to-one

Think of x as a "number" $\qquad$ Think of y as an "angle"

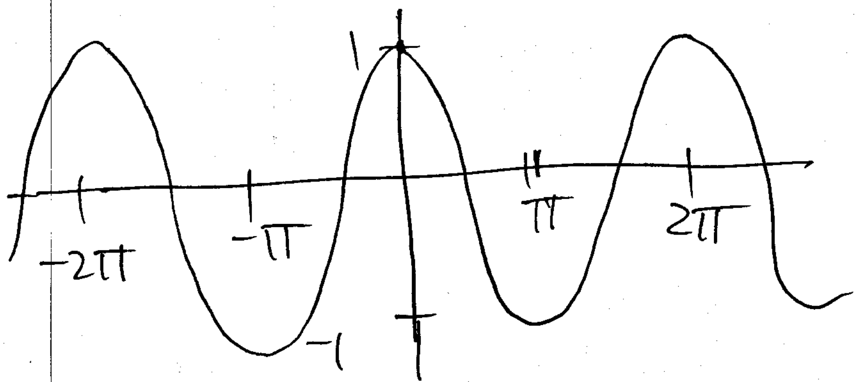
e.g. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ bec. $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin^{-1}(-1) = -\frac{\pi}{2}$ bec $\sin\left(-\frac{\pi}{2}\right) = -1$

but also $\sin\left(+\frac{3\pi}{2}\right) = -1$

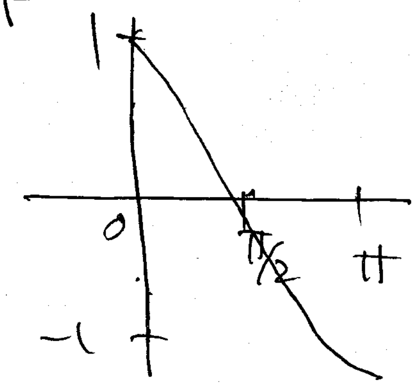
$\sin^{-1}(0) = 0$ $\qquad$ but $\frac{3\pi}{2} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\cos^{-1}(x) = \arccos(x)$



$\cos(x)$

restrict $\longrightarrow$



$y = \cos^{-1}(x) \iff \cos(y) = x$

$-1 \leq x \leq 1 \qquad\qquad 0 \leq y \leq \pi$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \longrightarrow \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \longrightarrow \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$

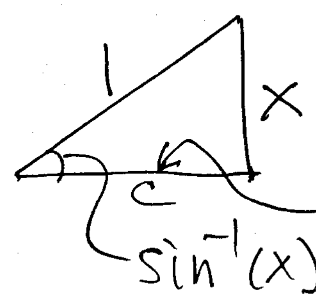
$$\boxed{\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{(1-x^2)^{1/2}}}$$

$$\sin(\sin^{-1}(x)) = x$$

$$\frac{d}{dx}\sin(\sin^{-1}(x)) = \frac{d}{dx}(x)$$

$$\cos(\sin^{-1}(x)) \cdot \frac{d}{dx}\sin^{-1}(x) = 1$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{(1-x^2)^{1/2}}$$



$$c^2 + x^2 = 1$$

$$c^2 = 1 - x^2$$

$$c = (1-x^2)^{1/2}$$

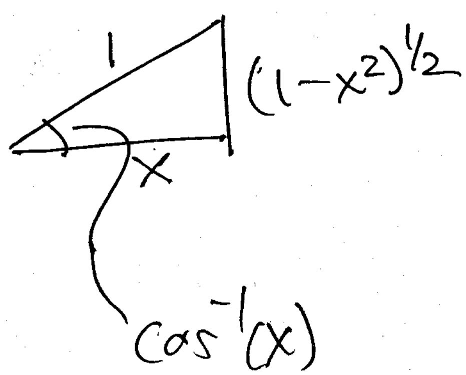
$$\cos(\sin^{-1}(x)) = (1-x^2)^{1/2}$$

$$\boxed{\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{(1-x^2)^{1/2}}}$$

$$\cos\left(\cos^{-1}(x)\right) = x$$

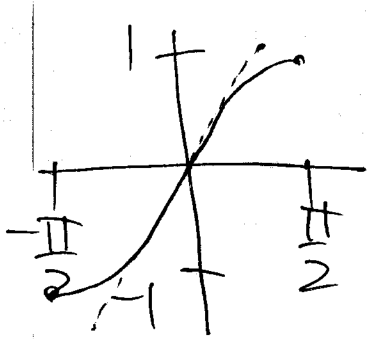
$$-\sin\left(\cos^{-1}(x)\right) \cdot \frac{d}{dx}\cos^{-1}(x) = 1$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{(1-x^2)^{1/2}}$$



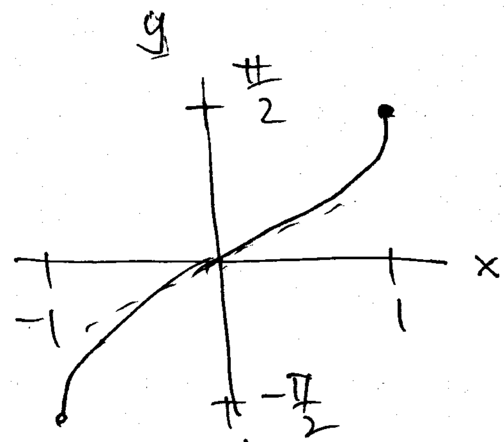
$$\sin(\cos^{-1}(x)) = (1-x^2)^{1/2}$$

$\sin^{-1}(x)$



$\sin(x)$

$\longrightarrow$



$\sin^{-1}(x)$

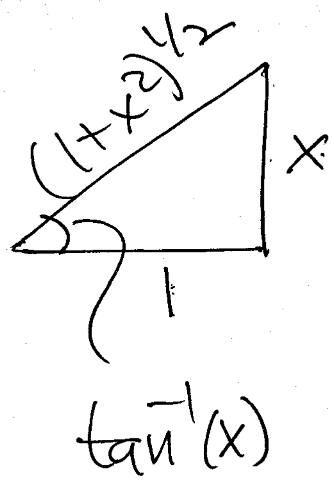
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{(1-x^2)^{1/2}}$$

$$\boxed{\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

$$\tan(\tan^{-1}(x)) = x$$

$$\sec^2(\tan^{-1}(x)) \cdot \frac{d}{dx}\tan^{-1}(x) = 1$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1+x^2}$$



$$\sec(\tan^{-1}(x)) = \frac{1}{\cos(\tan^{-1}(x))} =$$

$$= \frac{1}{\frac{1}{(1+x^2)^{1/2}}} = (1+x^2)^{1/2}$$

e.g. $f(t) = \ln\left(\tan^{-1}(t)\right)$

$$f'(t) = \frac{1}{\tan^{-1}(t)} \cdot \frac{1}{1+t^2}$$

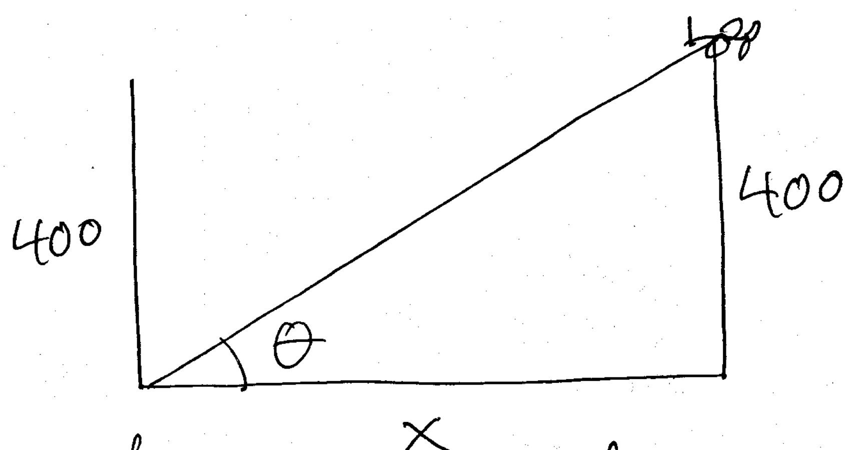
eg $f(x) = x\sin^{-1}(e^x)$

$$f'(x) = x \cdot \left[\frac{1}{(1-(e^x)^2)^{1/2}} \cdot e^x\right] + \sin^{-1}(e^x)$$

$$= \frac{xe^x}{(1-e^{2x})^{1/2}} + \sin^{-1}(e^x).$$

eg #30

Relate θ and x:



$$\tan(\theta) = \frac{400}{x}$$

$$\theta = \tan^{-1}\left(\frac{400}{x}\right)$$

$400x^{-1} \xrightarrow{\frac{d}{dx}} -400x^{-2}$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{400}{x}\right)^2} \cdot \frac{-400}{x^2}$$

$$= \frac{-400}{x^2+160000} \ \frac{rad}{m}$$