Recap: Rules for Differentiation

Product Rule

Quotient Rule

Trig Functions

Exponential Function $y = e^x$

Chain Rule ← Very important

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x).$$

$$u = g(x), \quad y = f(u) \longrightarrow y = f(g(x)) = (f \circ g)(x)$$

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(u)} \cdot \underbrace{\frac{dy}{dx}}_{g'(x)}$$

$$f'(u) \longleftrightarrow f'(g(x)).$$

① $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$

② $\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x).$

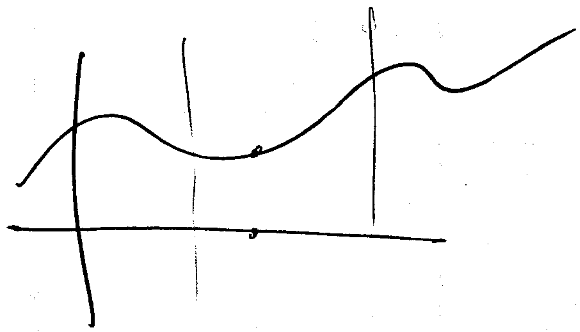
~~Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ if n integer~~

~~$\frac{d}{dx}(x^{\frac{m}{n}}) = \frac{d}{dx}[$~~

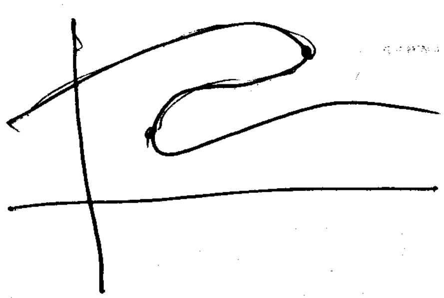
3.7 Implicit Differentiation.

Idea: How do we define curves?

① $y = f(x) \leftarrow$ graph is a curve

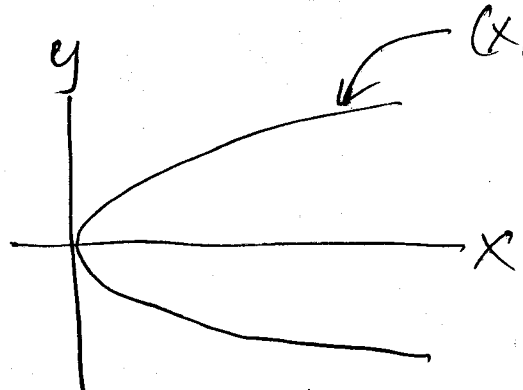


② Not every curve is graph of a function.

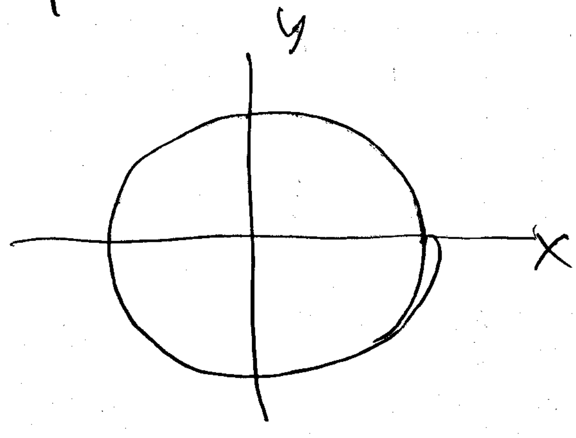


Any curve is the graph of some equation involving x and y.

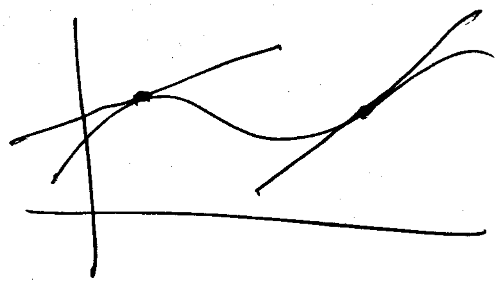
e.g. $x = y^2$



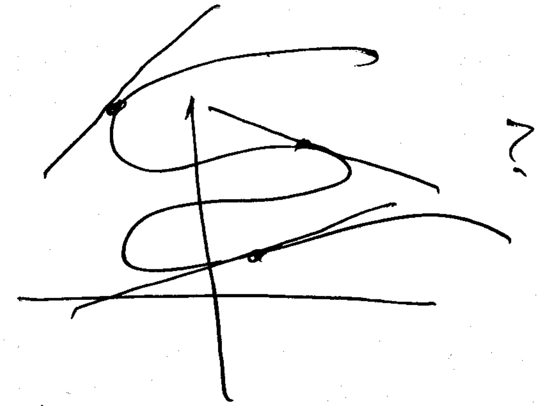
$x^2 + y^2 = 1$



Q: How do we find tangent lines to curves?



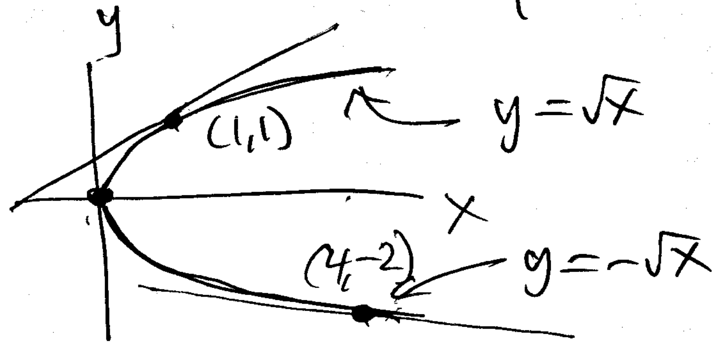
$y = f(x)$



Two solutions:

① $x = y^2$

why not graph of function?



Solve for y: $x = y^2$

$y = \pm\sqrt{x}$ two solutions

So curve is two functions:

$y = \sqrt{x}$ if $y \geq 0$ $\qquad$ $y = -\sqrt{x}$ if $y < 0$

Say I want slope of tangent line at $(1,1)$.

$$\frac{d}{dx}(\sqrt{x})\Big|_{x=1} = \frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2} \text{ slope.}$$

What about at $(4,-2)$?

$$\frac{d}{dx}(-\sqrt{x})\Big|_{x=4} = \frac{-1}{2\sqrt{x}} = \frac{-1}{4}$$

Problem is: sometimes impossible to solve for y in terms of x.

② Better solution $\rightarrow$ implicit differentiation

e.g. $x = y^2$

Think of y as some function of x

Take derivative of both sides w.r.t. x using chain rule.

$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$

$$1 = 2y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

$$\frac{d}{dx}[y(x)]^2 = 2\,y(x)\cdot\frac{dy}{dx}$$

$$x = [y(x)]^2$$

$$\frac{d}{dx}(x) = \frac{d}{dx}[y(x)]^2$$

$$1 = 2\cdot y(x)\cdot\frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{2}$$

$$\left.\frac{dy}{dx}\right|_{(4,-2)} = \frac{1}{2(-2)} = \frac{-1}{4}$$

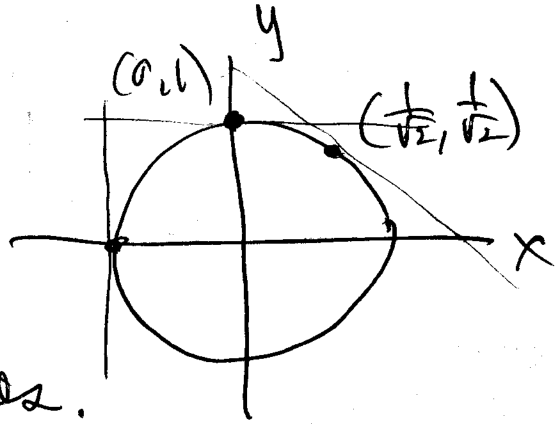
eg $x^2+y^2=1$. Find slope of tangent at $(0,1)$ $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $(-1,0)$

Trick:

Think of y as a function of x.

Take derivative w.r.t. x of both sides.

Use chain rule.



$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(0,1)}=-\frac{0}{1}=0$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}=\frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}=-1$$

$$\left.\frac{dy}{dx}\right|_{(-1,0)}=\frac{1}{0}$$ undefined

vertical tangent

Find $\frac{dy}{dx}$ if $x^3+y^3=18xy$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(18xy)$$

$$3x^2+3y^2\frac{dy}{dx} = 18\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)$$

$$3x^2+3y^2\frac{dy}{dx} = 18x\frac{dy}{dx}+18y$$

$$x^2+y^2\frac{dy}{dx} = 6x\frac{dy}{dx}+6y$$

$$y^2\frac{dy}{dx}-6x\frac{dy}{dx} = 6y-x^2$$

$$\frac{dy}{dx}(y^2-6x) = 6y-x^2$$

$$\frac{dy}{dx} = \frac{6y-x^2}{y^2-6x}$$

Find $\frac{dy}{dx}$ if $x^2=\frac{x-y}{x+y}$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\left(\frac{x-y}{x+y}\right)$$

$$2x = \frac{(x+y)(1-dy/dx)-(x-y)\left(1+\frac{dy}{dx}\right)}{(x+y)^2}$$

$$2x(x+y)^2 = \left(\cancel{x} + y - x\frac{dy}{dx} - \cancel{y\frac{dy}{dx}}\right) - \left(\cancel{x} - y + x\frac{dy}{dx} - \cancel{y\frac{dy}{dx}}\right)$$

$$= 2y - 2x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x(x+y)^2 - 2y}{-2x} = \frac{x(x+y)^2 - y}{-x}$$ //

Find $\dfrac{d^2y}{dx^2}$ if $\sqrt{y} + xy = 1$

$$\frac{d}{dx}\left(y^{1/2} + xy\right) = \frac{d}{dx}(1)$$

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}y^{-1/2}\frac{dy}{dx} + x\frac{dy}{dx} + y\right) = \frac{d}{dx}(0)$$

$$\frac{1}{2}y^{-1/2}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{-1}{4}y^{-3/2}\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{1}{2}y^{-1/2}\frac{d^2y}{dx^2} + x\frac{d^2y}{dx^2} = \frac{1}{4}y^{-3/2}\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{4}y^{-3/2}\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}}{\frac{1}{2}y^{-1/2} + x}$$

Could solve for $\frac{dy}{dx}$ and plug in here

3.8 Derivative of log and exp functions

① $y = \ln(x)$ (review sec. 1.3 if needed)

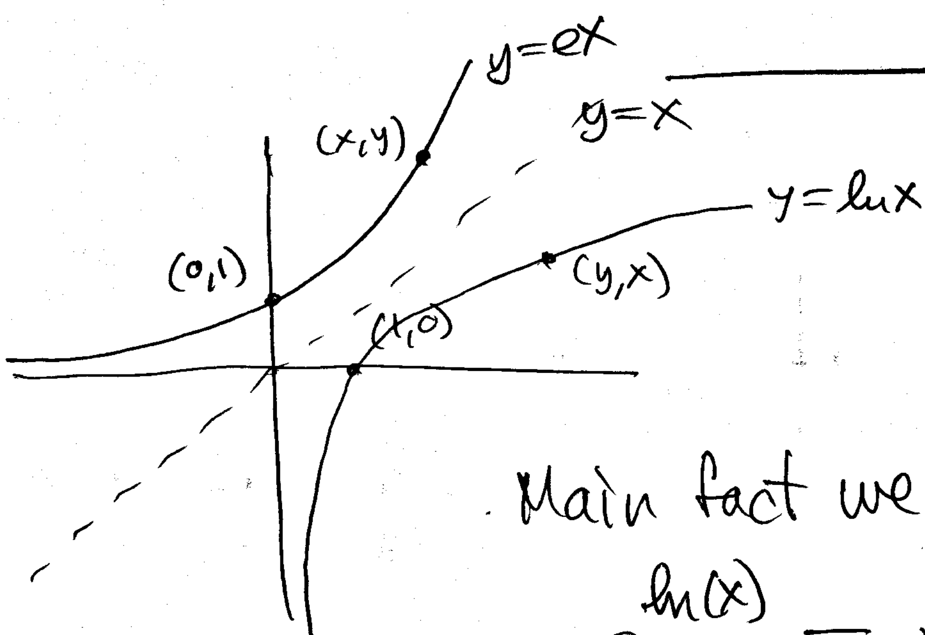
$\ln$ is the inverse function of e^x.

This means: $y = \ln(x) \iff e^y = x$

$e^y = x$, solve for y

$\hookrightarrow y = \ln(x)$

$e^y = 8 \qquad y = \ln(8)$



Main fact we need:

$$e^{\ln(x)} = x$$

$$\frac{d}{dx}\left(e^{\ln(x)}\right) = \frac{d}{dx}(x)$$

$$e^{\ln(x)} \cdot \frac{d}{dx}(\ln(x)) = 1$$

$$x \cdot \frac{d}{dx}\ln(x) = 1$$

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

Remember also laws of logarithms

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(x^a) = a\ln(x).$$

e.g. $\frac{d}{dx}(\ln(x^2)) = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$

or $\frac{d}{dx}(\ln(x^2)) = \frac{d}{dx}(2\ln(x)) = 2\cdot\frac{1}{x} = \frac{2}{x}$.

eg $\frac{d}{dx} \ln(2(x^2+1)) = \frac{1}{\cancel{2}(x^2+1)} \cdot \overset{2}{\cancel{4}}x$

$= \frac{2x}{x^2+1}$

or $\frac{d}{dx} \ln(2(x^2+1)) = \frac{d}{dx}\left(\ln(2) + \ln(x^2+1)\right)$

$= 0 + \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$