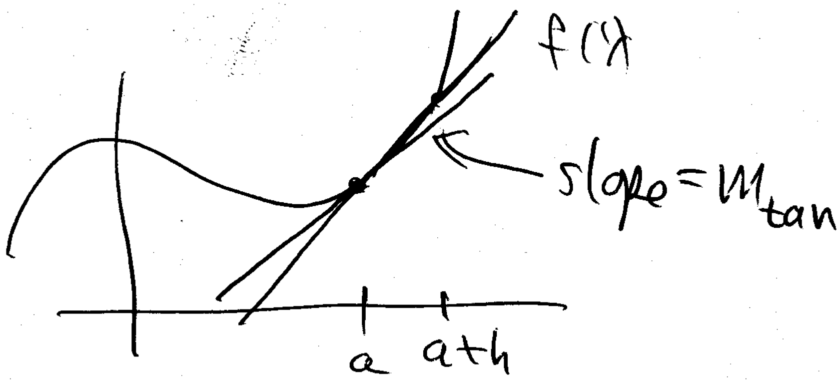
Quiz 3 2.5, 2.6 NO CALCULATORS

The Derivative

$y = f(x)$



$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

e.g $g(t) = \frac{8}{t^2}$ $a = 3$ Equ of tangent line at $a=3$.

Need $\underbrace{\text{a point}}$ & $\underbrace{\text{slope}}$

$(3, g(3)) = (3, \frac{8}{9})$

$$m_{tan} = \lim_{h \to 0} \frac{g(3+h) - g(3)}{h} = \lim_{h \to 0} \frac{\frac{8}{(3+h)^2} - \frac{8}{9}}{h}$$

$$\frac{1}{h}\left(\frac{8}{(3+h)^2} - \frac{8}{9}\right) = \frac{1}{h} \frac{72 - 8\overbrace{(3+h)^2}^{9+6h+h^2}}{9(3+h)^2} = \frac{\cancel{72} - \cancel{72} - 48h - 8h^2}{9h(3+h)^2}$$

$$= \frac{\cancel{h}(-48-8h)}{9\cancel{h}(3+h)^2} = \frac{-48-8h}{9(3+h)^2}$$

$$= \lim_{h\to 0} \frac{-48-8h}{9(3+h)^2} = \frac{-48}{81} = \frac{-16}{27} \leftarrow \text{slope}$$

Eqn of line: $y - \frac{8}{9} = \frac{-16}{27}(x-3)$

$$y = \frac{-16}{27}x + \frac{16}{9} + \frac{8}{9}$$

$$y = -\frac{16}{27}x + \frac{24}{9} = -\frac{16}{27}x + \frac{8}{3} \text{ //.}$$

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e.g. $h(t) = \sqrt{t+1}$ $\quad a=3$

point: $(3, h(3)) = (3,2)$

slope: $\lim_{h\to 0} \frac{\sqrt{(3+h)+1} - 2}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$

$$\frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(\cancel{4}+h)-\cancel{4}}{h(\sqrt{4+h}+2)} = \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$= \frac{1}{\sqrt{4+h}+2}, \underline{h \neq 0}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4} \leftarrow \text{slope}$$

~~y-2~~ Eqn of line: $y-2 = \frac{1}{4}(x-3)$

$$y = \frac{1}{4}x - \frac{3}{4} + 2 = \frac{1}{4}x + \frac{5}{4} \quad //$$

<u>eg</u> $f(x) = \frac{x}{x+1}$ $\quad a=1$

point: $(1, f(1)) = (1, \frac{1}{2})$

slope: $\lim_{h \to 0} \frac{\frac{1+h}{(1+h)+1} - \frac{1}{2}}{h} = \lim_{h \to 0} \frac{\frac{1+h}{2+h} - \frac{1}{2}}{h}$

$$\left(\frac{1+h}{2+h} - \frac{1}{2}\right) \cdot \frac{1}{h} = \frac{2+2h-(2+h)}{2(2+h)h} = \frac{\cancel{2}+2h-\cancel{2}-h}{2(2+h)h}$$

$$= \frac{\cancel{h}}{2(2+h)\cancel{h}} = \frac{1}{2(2+h)}, \quad h \neq 0$$

$$= \lim_{h \to 0} \frac{1}{2(2+h)} = \frac{1}{4}$$

Eqn of line:

$y - \frac{1}{2} = \frac{1}{4}(x-1)$

$y = \frac{1}{4}x + \frac{1}{4}$ //

The Derivative

Now think of slope of tangent line as a function of x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(i.e. we replace a by x in formula for m_{tan}.)

<u>e.g.</u> $f(x) = x^2$ $\quad f'(x) = 2x.$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

<u>eg</u> $g(t) = \frac{8}{t^2}$ $\quad g'(t) = \frac{-16}{t^3}$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{\frac{8}{(t+h)^2} - \frac{8}{t^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{8t^2 - 8(t+h)^2_{=t^2+2th+h^2}}{t^2(t+h)^2}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{\cancel{8t^2} - \cancel{8t^2} - 16th - 8h^2}{t^2(t+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-16t - 8h)}{\cancel{h}t^2(t+h)^2} = \lim_{h \to 0} \frac{-16t - 8h}{t^2(t+h)^2} = \frac{-16t}{t^4}$$

$$= \frac{-16}{t^3}. \quad \left(\text{So } g'(3) = \frac{-16}{(3)^3} = \frac{-16}{27}\right)$$

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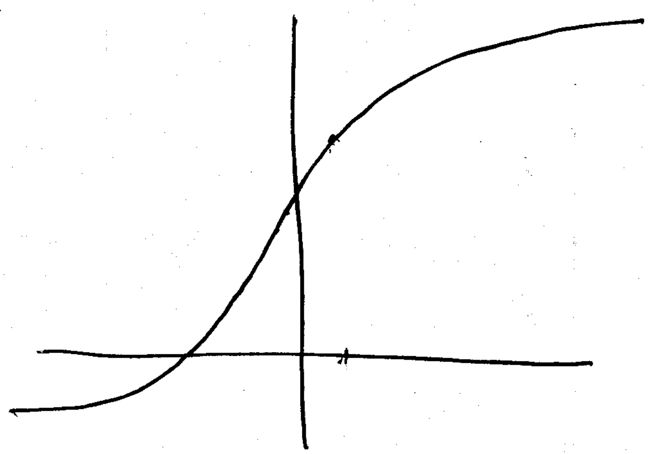
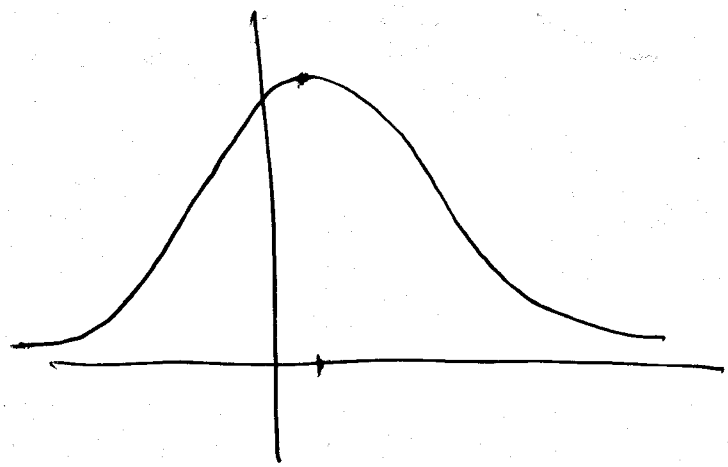


f



f'

19, 2.7)

$$\lim_{x \to 1} (8x+5) = 13$$

Idea: For any $\varepsilon > 0$, $\boxed{\text{find } \delta > 0}$ so that whenever $0 < |x-1| < \delta$, $\underbrace{|(8x+5)-13| < \varepsilon}$.

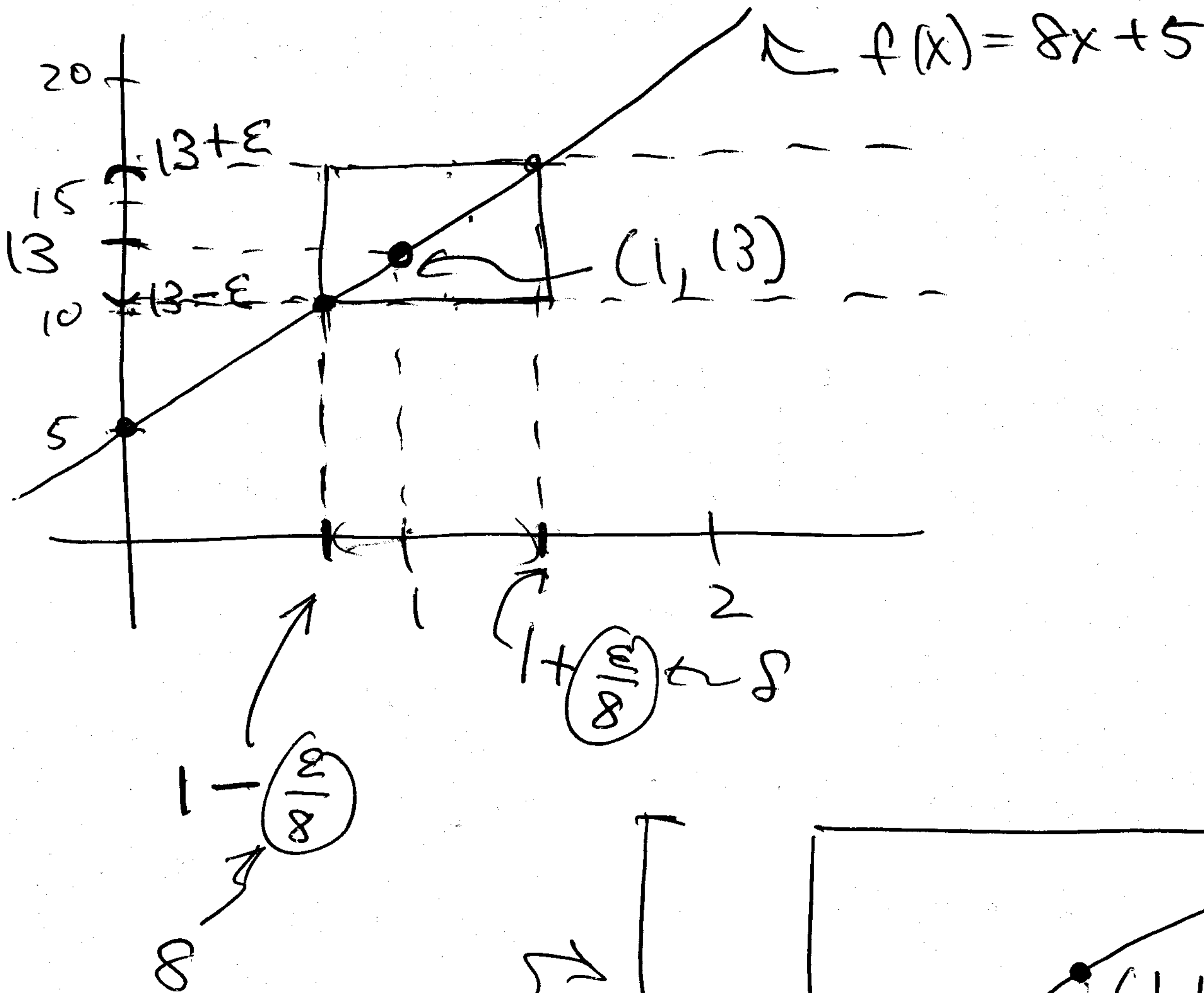
$$|8x+5-13| < \varepsilon$$
$$\Updownarrow$$
$$|8x-8| < \varepsilon$$
$$\Updownarrow$$
$$|8(x-1)| < \varepsilon$$
$$\Updownarrow$$
$$8|x-1| < \varepsilon$$
$$\Updownarrow$$
$$|x-1| < \frac{\varepsilon}{8}$$

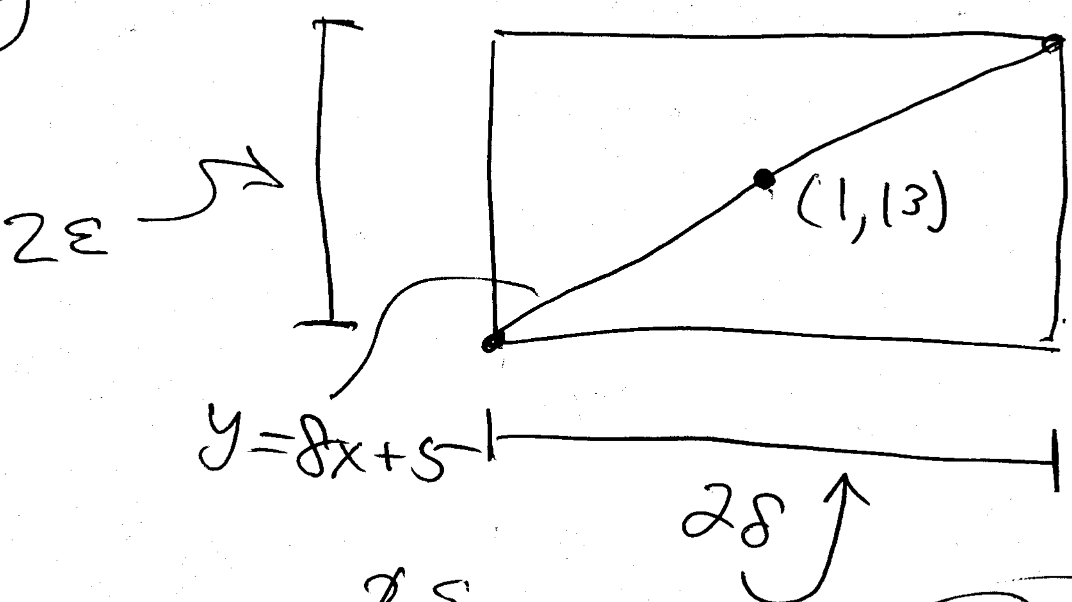
This means! If $0 < |x-1| < \frac{\varepsilon}{8}$ then $|(8x+5)-13| < \varepsilon$

So we can take $\boxed{\delta = \frac{\varepsilon}{8}}$

$\varepsilon = \frac{1}{10}$ $\delta = \frac{1}{80}$

$\varepsilon = \frac{1}{50}$ $\delta = \frac{1}{400}$

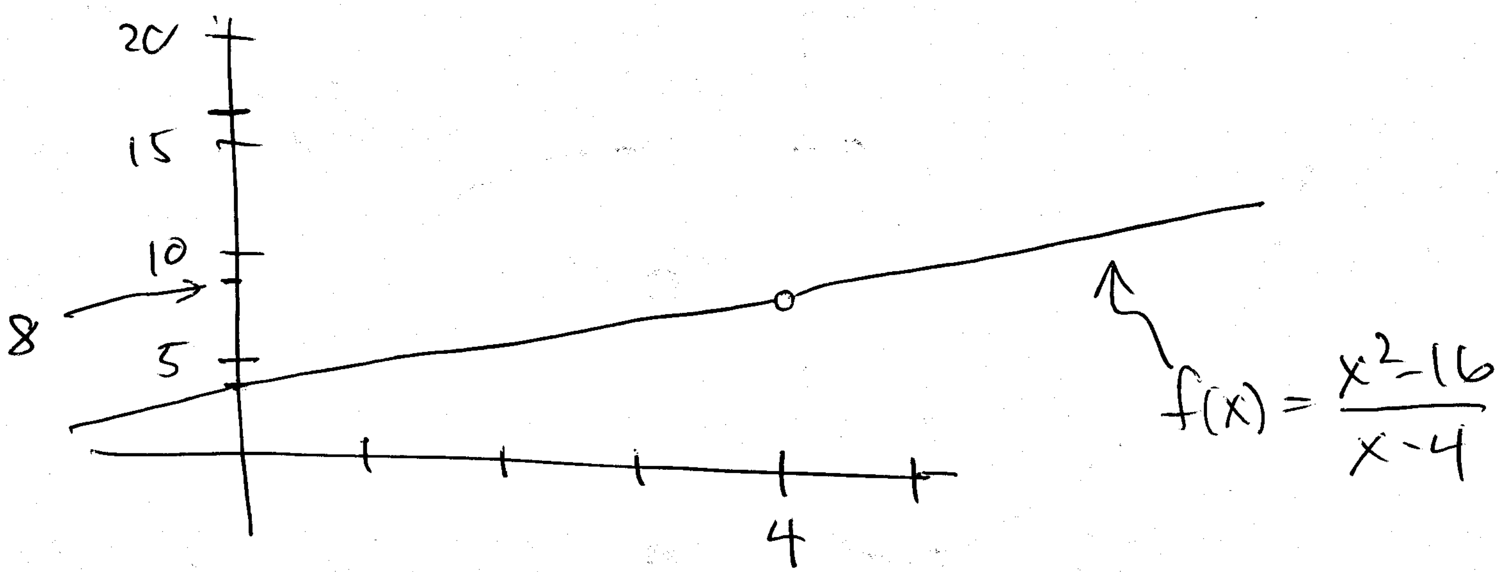




slope = 8 means $\frac{\cancel{2}\varepsilon}{\cancel{2}\delta} = 8 \longrightarrow \delta = \frac{\varepsilon}{8}$

21) $\lim\limits_{x \to 4} \dfrac{x^2-16}{x-4} = 8$

$$\boxed{\frac{x^2-16}{x-4} = \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}} = x+4 \text{ if } x \neq 4}$$



For any $\varepsilon > 0$, find $\delta > 0$ so that whenever $0 < |x-4| < \delta$, $\left|\dfrac{x^2-16}{x-4} - 8\right| < \varepsilon$

$\left|\dfrac{x^2-16}{x-4} - 8\right| < \varepsilon$ $\rightarrow$ So if $0 < |x-4| < \varepsilon$

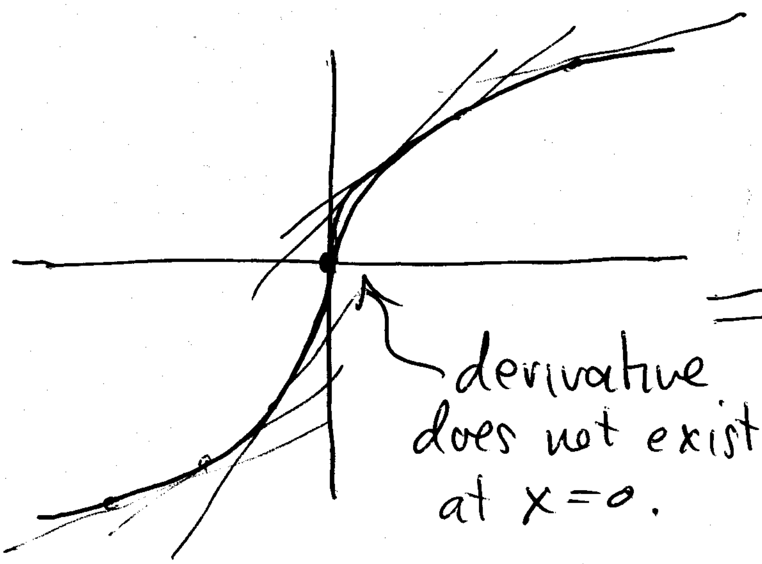
then $\left|\dfrac{x^2-16}{x-4} - 8\right| < \varepsilon$

$|x+4-8| < \varepsilon$

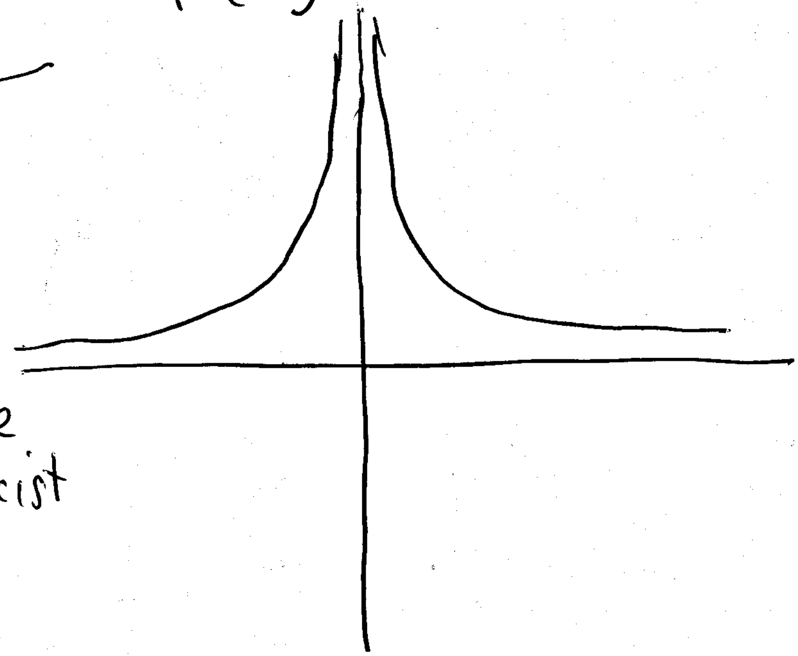
$|x-4| < \varepsilon$

that is, take $\boxed{\delta = \varepsilon}$

e.g. $f(x) = x^{1/3}$ $\qquad$ $f'(x) =$



$f(x)$



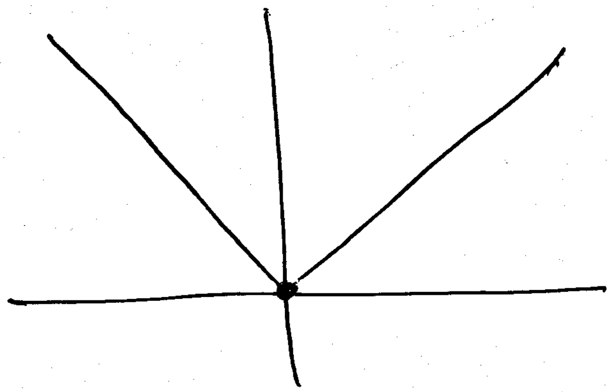
$f'(x)$

$f'(0)$ does not exist

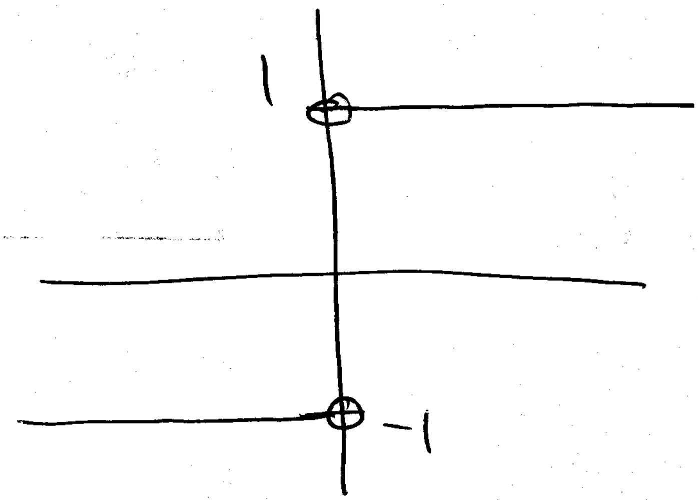
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

$$= \infty$$

$f(x) = |x|$



$f(x)$



$f'(x)$.

$f'(0)$ does not exist.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} \text{ does not exist}$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

3.2 Rules of Differentiation

① Polynomials.

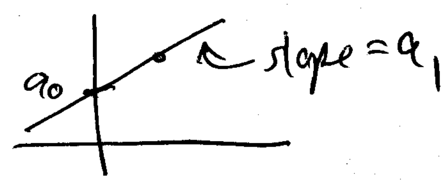
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$-\ f(x) = a_0$ (so const function)

$f'(x) = 0$

$-\ f(x) = a_0 + a_1 x$ (linear function)

$f'(x) = a_1 \leftarrow$ slope of line



$-\ f(x) = a_0 + a_1 x + a_2 x^2$

$\leftarrow$ derivative $= 2a_2 x$

$g(x) = a_2 x^2$

$$\lim_{h \to 0} \frac{a_2(x+h)^2 - a_2 x^2}{h} = a_2 \underbrace{\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}}_{2x}$$

$-\ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Rule: If n is a positive integer, $n = 1, 2, 3, \ldots$

$$f(x) = x^n \longrightarrow f'(x) = n x^{n-1}$$

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{\cancel{x^n} + nx^{n-1}\cancel{h} + (\,)h^2 + (\,)h^3 + \cdots (\,)h^n \cancel{-x^n}}{\cancel{h}}$$

$$\boxed{(x+h)^n = x^n + nx^{n-1}h^1 + (\,)x^{n-2}h^2 + (\,) \cdots + (\,)x^1h^{n-1} + h^n}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

$1 \quad 2 \quad 1 \longrightarrow (x+h)^2 = x^2 + 2xh + h^2$

$1 \quad 3 \quad 3 \quad 1 \longrightarrow (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$1 \quad 4 \quad 6 \quad 4 \quad 1 \longrightarrow (x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$$= \lim_{h\to 0} nx^{n-1} + \underbrace{(\,)h + (\,)h^2 + \cdots + (\,)h^{n-1}}_{0}$$

$$= nx^{n-1}$$