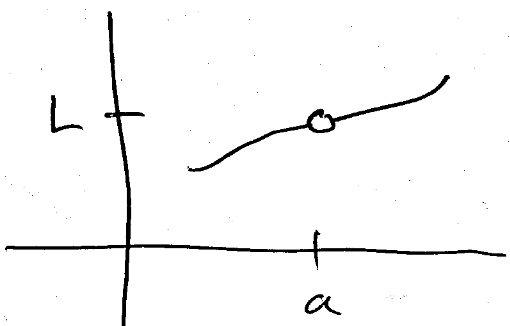
Quiz 3 2.5, 2.6

2.7 Precise Def'n of Limit

Idea: We already have an informal definition of limit (or could say intuitive definition) $\lim_{x \to a} f(x) = L$



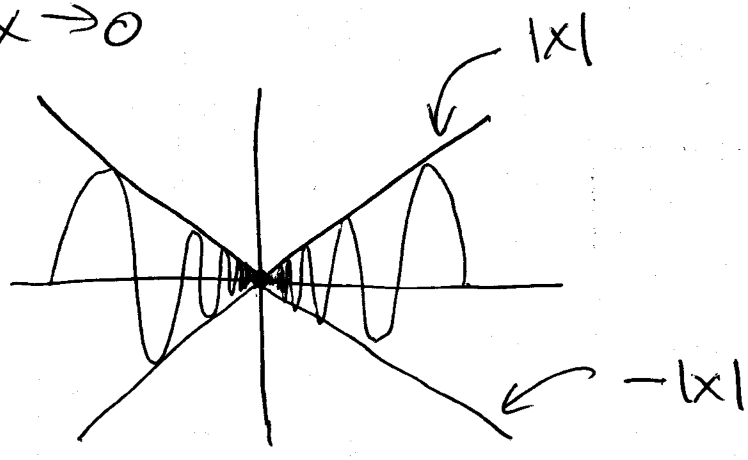
graph of f passes through (a, L).

Evaluating f(x) for x nearer and nearer to a gives f(x) nearer and nearer to L.

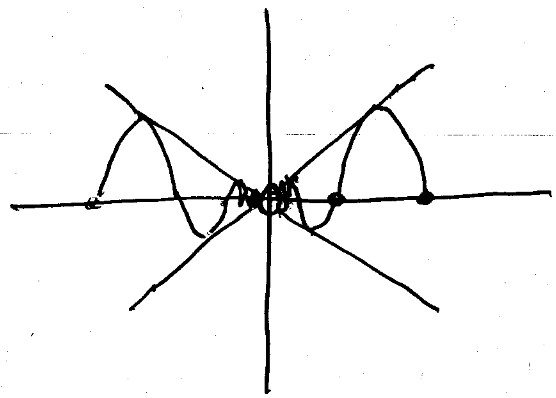
This definition is fine for almost all circumstances.

e.g. Intuition is a little unclear for

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

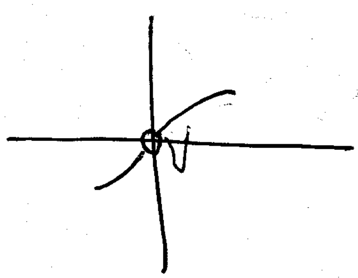


Does $x\sin\left(\frac{1}{x}\right)$ get steadily closer and closer to 0 as x gets closer to zero?

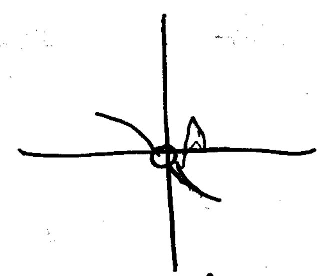


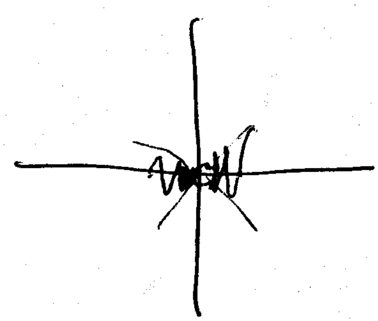
No. Keeps going away from zero, then back again.

Does $x\sin\left(\frac{1}{x}\right)$ pass through $(0,0)$? <u>NO</u>



or





never does either of these.

<u>Definition</u>: $\lim_{x\to a} f(x) = L$ means that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

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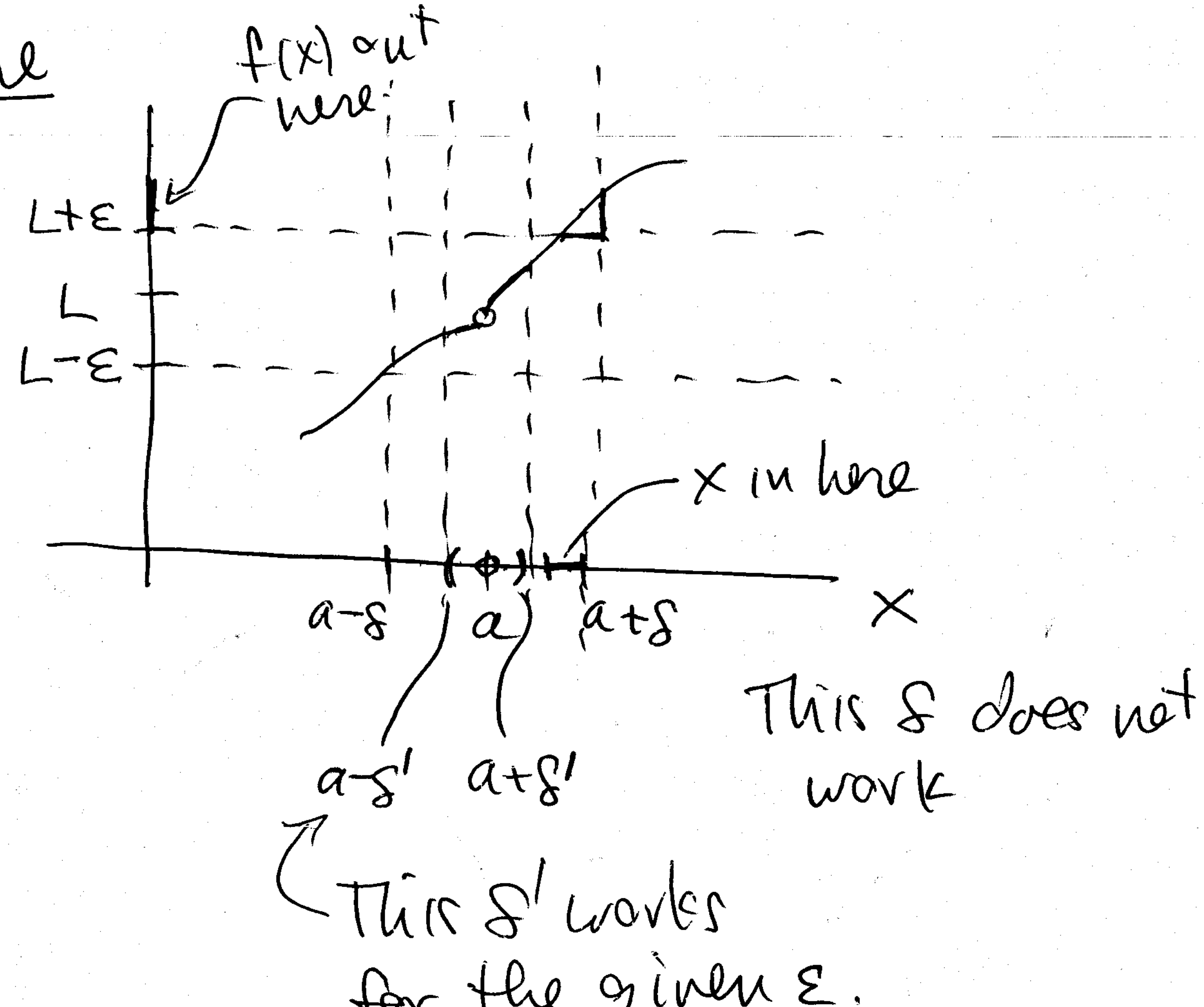
① Think of ε as an <u>error tolerance</u>

② Think of δ as a <u>sensitivity</u> of f.

How close do I have to be to a to guarantee $f(x)$ is "within tolerance".

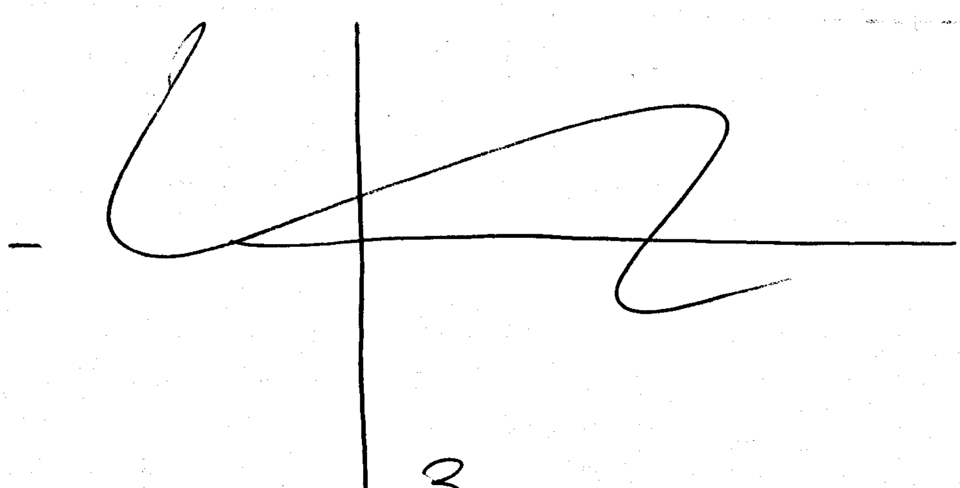
③ $0<|x-a|<\delta$ says x is within δ of a <u>and</u> <u>$x \neq a$</u>.

<u>Picture</u>

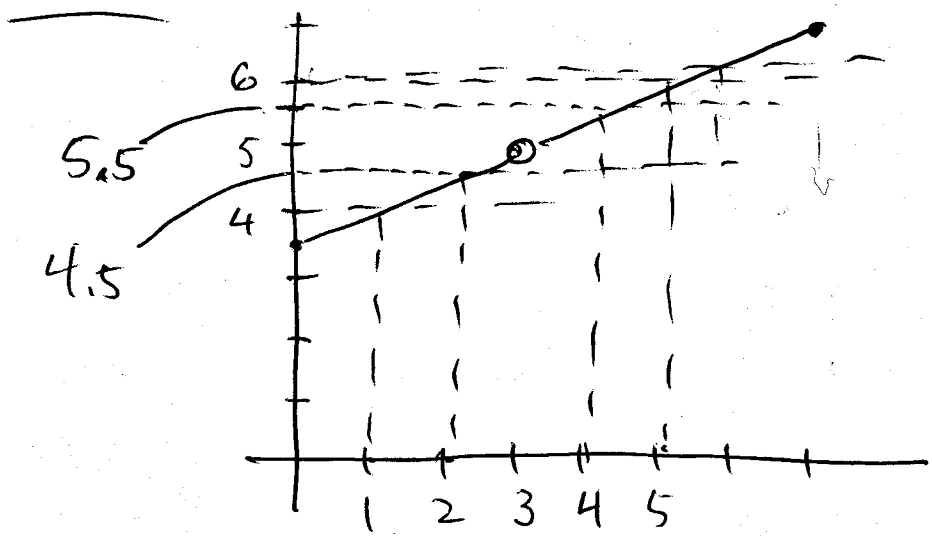


Question is: Given $\varepsilon > 0$, can I find a $\delta > 0$ that works?

<u>EG 1</u>



EG 1



$\varepsilon = 1 \Leftrightarrow \delta = 2$
or smaller

$\varepsilon = \frac{1}{2} \Leftrightarrow \delta = 1$
or
$\delta < 1$
works
also.

Notice: slope of this line is $\frac{1}{2}$.

and equation is $f(x) = \frac{1}{2}x + \frac{7}{2}$

Any $\varepsilon > 0 \Longleftrightarrow \delta = \frac{\varepsilon}{1/2} = 2\varepsilon$

works.

#11) a. $\varepsilon = 3 \Longleftrightarrow \delta = 2 \; (\delta < 2)$

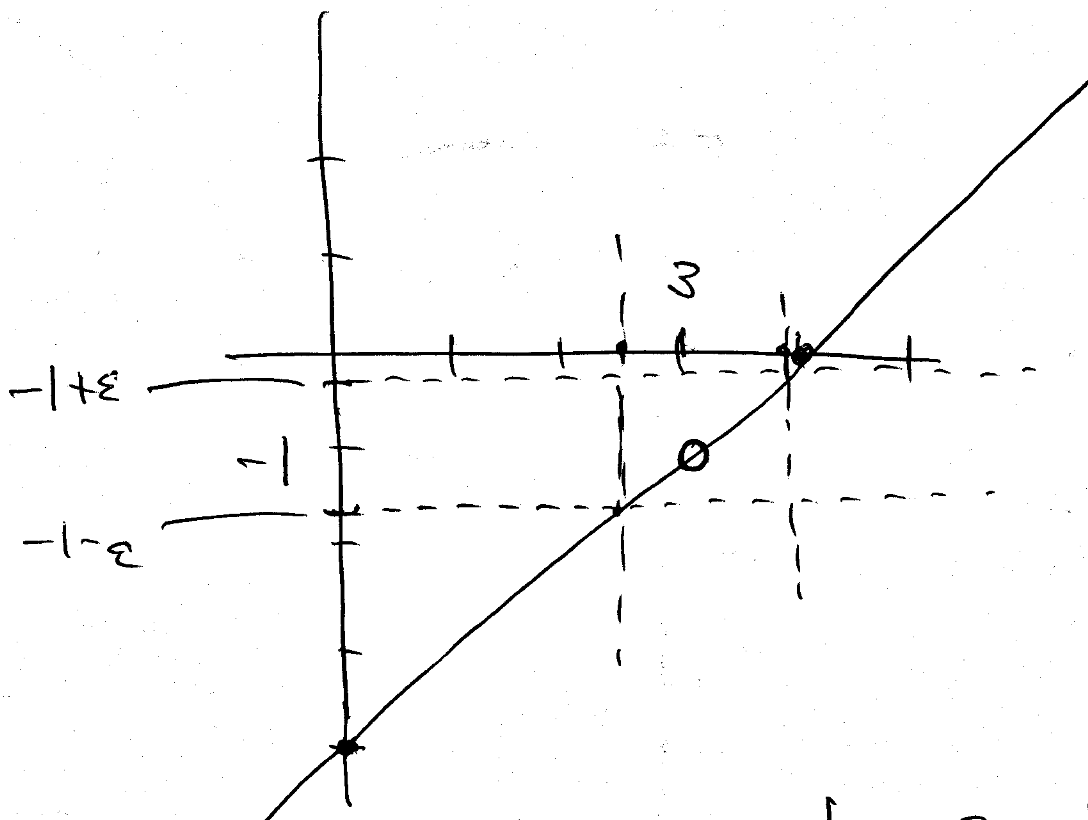
b. $\varepsilon = 1 \Longleftrightarrow \delta = \frac{1}{2} \; \left(\delta < \frac{1}{2}\right)$

#15) $\varepsilon = 1 \Longleftrightarrow \delta = 1$ (or $\delta < 1$)

#22) $\lim_{x \to 3} \left(\frac{x^2-7x+12}{x-3} \right) = -1$

$$\frac{x^2-7x+12}{x-3} = \frac{(x-4)\cancel{(x-3)}}{\cancel{x-3}} = x-4$$

if $x \neq 3$



Given $\varepsilon > 0$, can choose $\delta = \varepsilon$.

OR Want $|f(x) - (-1)| < \varepsilon$

$|x-4-(-1)| < \varepsilon$

$\underbrace{|x-3|}_{\delta} < \varepsilon$ so $\delta = \varepsilon$ works

2.5 40)

$$f(x) = \frac{\sqrt{16x^4+64x^2} + x^2}{2x^2-4}$$

$5x^2$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{((16x^4+64x^2)^{1/2}+x^2)}{2x^2-4}$$

$2x^2$ is dom. term

$$\boxed{(16x^4+64x^2)^{1/2} \overset{?}{\neq} 4x^2+8x}$$

No! No! No! No!

$$= \lim_{x\to\infty} \frac{5\cancel{x^2}}{2\cancel{x^2}} = \frac{5}{2}.$$

$$\frac{\left[(16x^4+64x^2)^{1/2}+x^2\right]}{\left[2x^2-4\right]} \frac{\frac{1}{x^2}}{\left(\frac{1}{x^2}\right)}$$

$$(16x^4+64x^2)^{1/2} \cdot \frac{1}{x^2} = \left(\frac{16x^4+64x^2}{x^4}\right)^{1/2} = \left(16+\frac{64}{x^2}\right)^{1/2}$$

$$= \frac{\left(16+\cancel{\frac{64}{x^2}}^{\,0}\right)^{1/2}+1}{2-\cancel{\frac{4}{x^2}}^{\,0}}$$

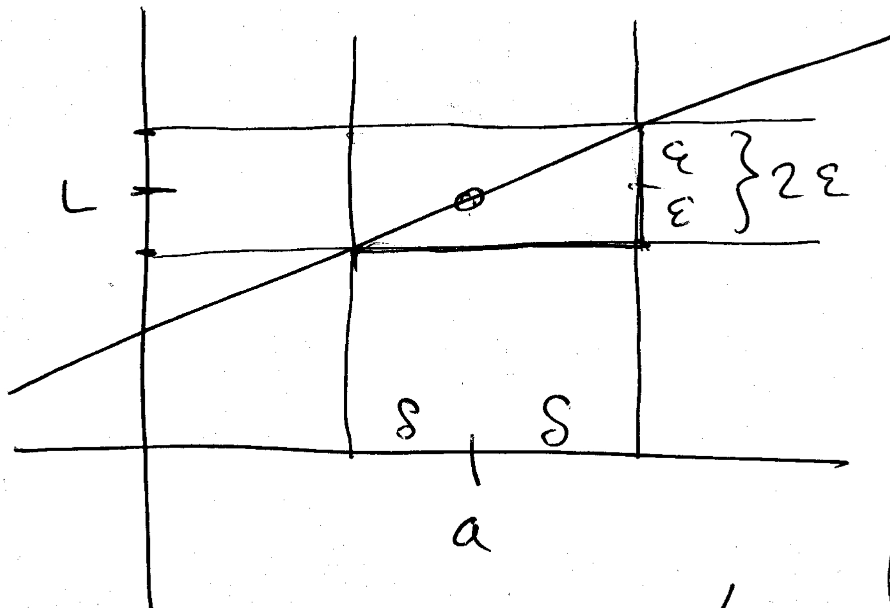
$$\xrightarrow{x\to\infty} \frac{16^{1/2}+1}{2} = \frac{4+1}{2} = \frac{5}{2}.$$

vertical asymp:

$$2x^2-4=0$$

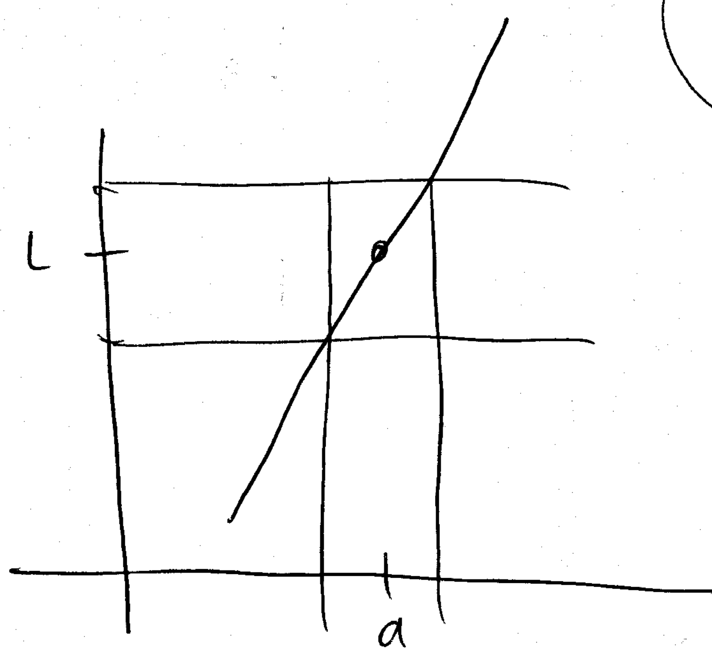
$$x^2-2=0$$

$$x=\pm\sqrt{2}. \leftarrow \text{vert asy.}$$



$$\cancel{2}\delta \times \text{slope} = \cancel{2}\varepsilon$$

$$\delta = \frac{\varepsilon}{\text{slope}}$$



3.1 Introduction to the Derivative

We have seen 2 equivalent concepts:

① Instantaneous rate of change of $y=f(x)$ at $x=a$.

- First compute average rates of change over intervals $[a,a+h]$

$$\frac{\Delta y}{\Delta x} = av_{[a,a+h]} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

- For instantaneous r.o.c. take limit

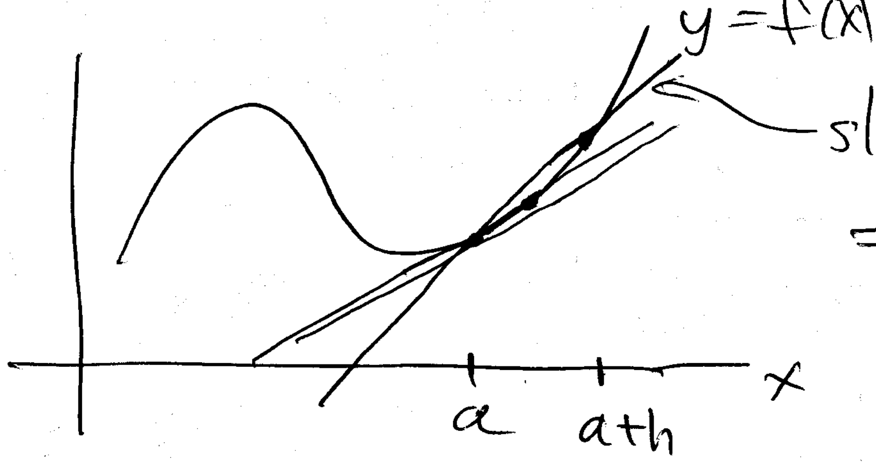
$$inst_{x=a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Note: $\frac{f(a+h)-f(a)}{h}$ evaluates to $\frac{0}{0}$ at $h=0$

Equivalently we could write

$$inst_{x=a} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

② Slope of tangent line to graph of $y=f(x)$ when $x=a$.



slope $= m_{sec}$

$= \dfrac{f(a+h)-f(a)}{h} = \dfrac{\Delta y}{\Delta x}$

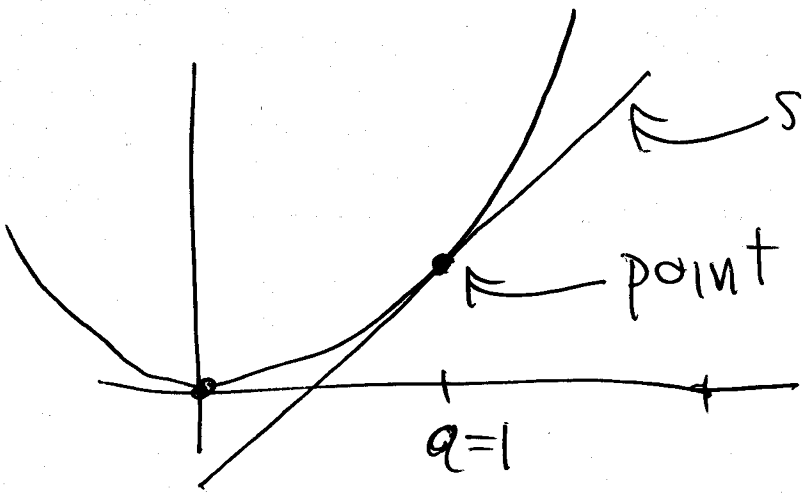
$$m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Equivalently

$$m_{tan} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

e.g. $f(x) = x^2$ $a = 1$

Find slope/equation of tangent line.



$= (1, f(1)) = (1, 1)$

Find $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2-1}{h}$

$$\frac{(1+h)^2-1}{h} = \frac{\cancel{1}+2h+h^2-\cancel{1}}{h} = \frac{\cancel{h}(2+h)}{\cancel{h}} = 2+h$$

$h \neq 0$

$= \lim_{h\to 0}(2+h) = \underline{2} \leftarrow$ slope

Eqn of line: $y - 1 = 2(x-1)$

m (x_0, y_0)

$y - y_0 = m(x - x_0)$

$y = 2x - 2 + 1$

$\boxed{y = 2x - 1}$