2.1 Idea of Limits

⊗ Differential Calculus
Integral Calculus

Two fundamental concepts of diff. calc.
① instantaneous velocity (or rate of change)
② the limit.

---

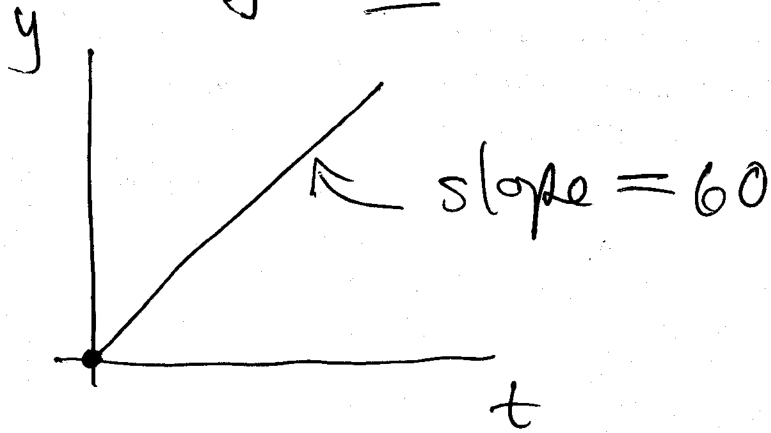
① How to find velocity?

$$\text{velocity} = \frac{\text{distance}}{\text{time}}$$

e.g. y = trip odometer in miles
t = elapsed time from start in hours.

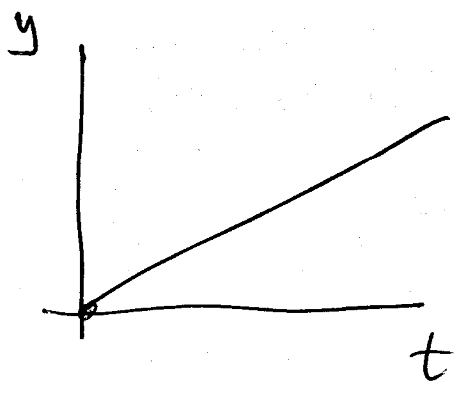
Assume constant velocity (say 60 m/hr)

Then $y = \underline{60}t$



e.g. Suppose speedometer broken but odometer + watch work.

How do I find velocity (assume constant)



$$\boxed{y = mt}$$

$\uparrow$ slope (= velocity)

a. Choose 2 times t_1, t_2

elapsed time $= \Delta t = t_2 - t_1$

b. Read odometer at t_1, t_2 and get say y_1, y_2 so $\Delta y = y_2 - y_1$

c. velocity $= m = \dfrac{\Delta y}{\Delta t} = \dfrac{y_2 - y_1}{t_2 - t_1}$

$$= \frac{mt_2 - mt_1}{t_2 - t_1} = \frac{m(\cancel{t_2 - t_1})}{\cancel{t_2 - t_1}} = m.$$

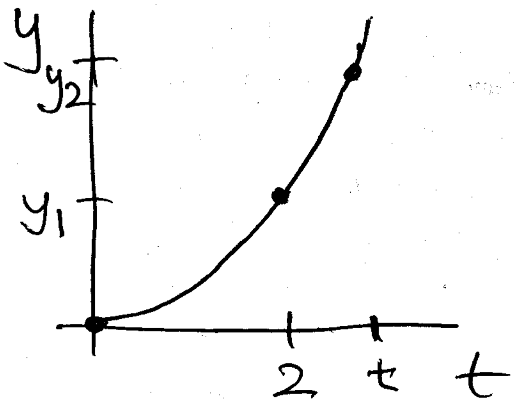
e.g. Suppose velocity not constant

e.g. free fall

y = dist a dropped object has fallen in feet

t = time since dropped in seconds.

$$y = 16t^2$$



not a line.

What is the slope?

Want velocity of object at $t=2$.

Find average velocity between $t_2 = t$ and $t_1 = 2$

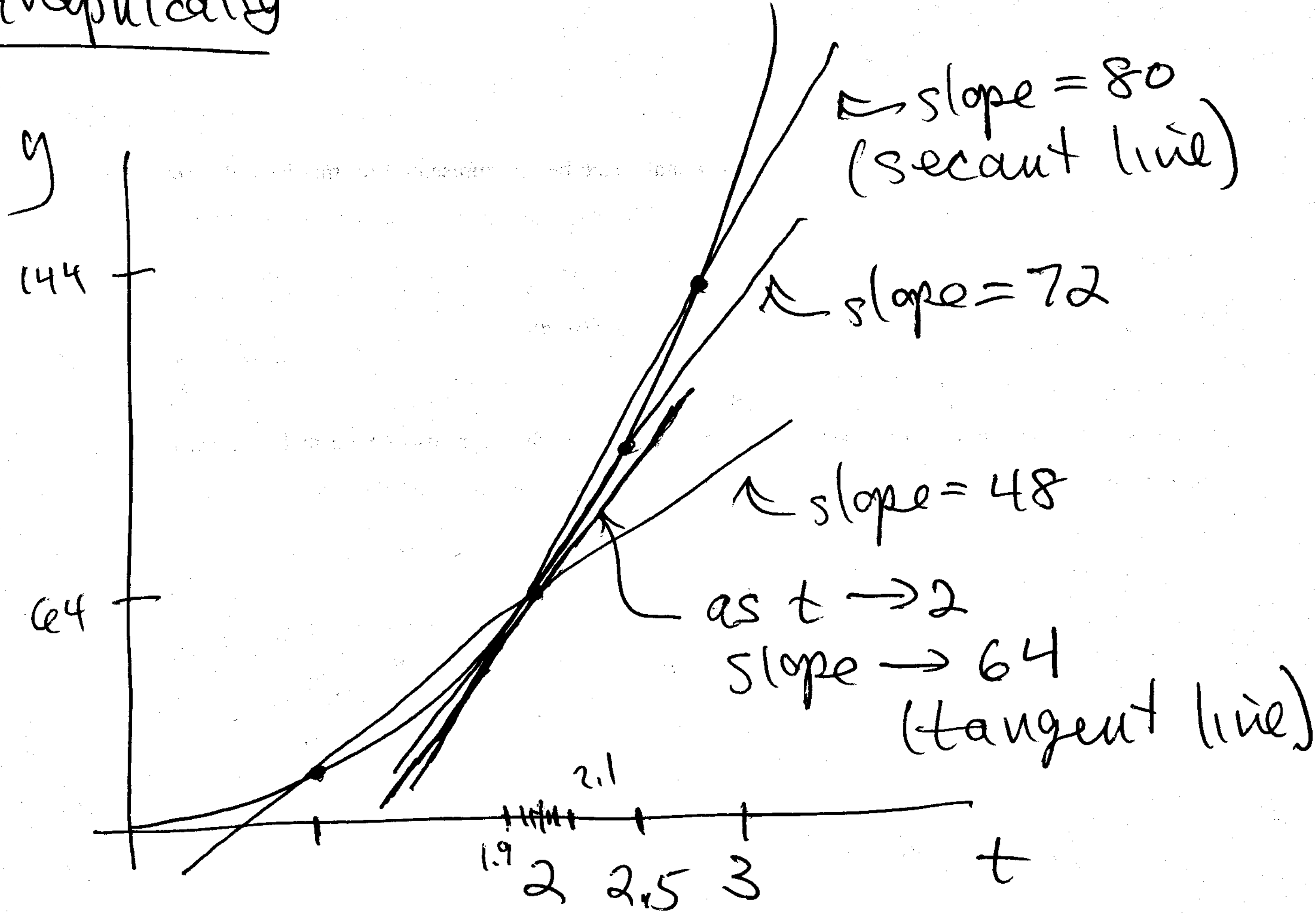
$$\text{avg vel} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t} = \frac{16t^2 - 64}{t-2} = 16 \cdot \frac{t^2-4}{t-2}$$

$$y_2 = 16t_2^2 \quad y_1 = 16t_1^2 = 16(2)^2 = 64$$

Idea: Take $t = t_2$ close to $t_1 = 2$ but not equal to it.

t	$16 \cdot \frac{t^2-4}{t-2}$
3	$16 \cdot \frac{5}{1} = 80$ ft/sec.
2.5	$16 \cdot \frac{6.25-4}{.5} = 16(4.5) = 72$ f/s
1.5	56 f/s
1	48 f/s
2.1	$16 \cdot \frac{(2.1)^2-4}{2.1-2} = 65.6$ f/s
1.9	62.4 f/s
2.05	64.8 f/s
1.95	63.2 f/s
2.01	64.16 f/s
1.99	63.84 f/s
$\downarrow$	$\downarrow$
2 sec.	64 $\frac{\text{feet}}{\text{sec}}$

Graphically



eg #9 $s(t) = -16t^2 + 128t$

$[1, 2]$

$$\frac{s(2) - s(1)}{2 - 1} = \frac{-64 + 256 - (-16 + 128)}{1} = \frac{192 - 112}{1} = 80$$

$[1, 1.5]$

$$\frac{s(1.5) - s(1)}{1.5 - 1} = \frac{(-16(1.5)^2 + 128(1.5)) - 112}{.5} = \frac{-36 + 192 - 112}{.5} = \frac{44}{.5} = 88$$

$[1, 1.001]$

$$\frac{s(1.001) - s(1)}{1.001 - 1} = \frac{-16.032016 + 128.128 - 112}{.001} = 95.984$$

2.2 Definition of Limit

$$\lim_{x \to a} f(x) = L$$

e.g. $$\lim_{t \to 2} 16\left(\frac{t^2-4}{t-2}\right) = 64$$

Idea: ① For calculus limits are used (mostly) to give meaning to expressions that ~~eva~~ evaluate to $\frac{0}{0}$

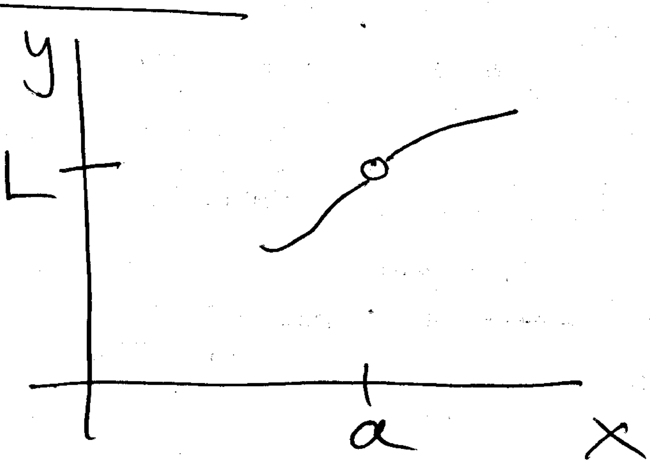
② $\lim_{x \to a} f(x)$ depends on values $f(x)$ for x near a but not equal to a

③ $\lim_{x \to a} f(x)$ does not have to exist.

Graphically

$y = f(x)$

$\lim_{x \to a} f(x) = L$



Point:

$\lim_{x \to a} f(x)$ has nothing to do with $f(a)$.

Numerically

$$\lim_{t\to 2} \underbrace{16\cdot\frac{t^2-4}{t-2}}_{f(t)} = 64$$

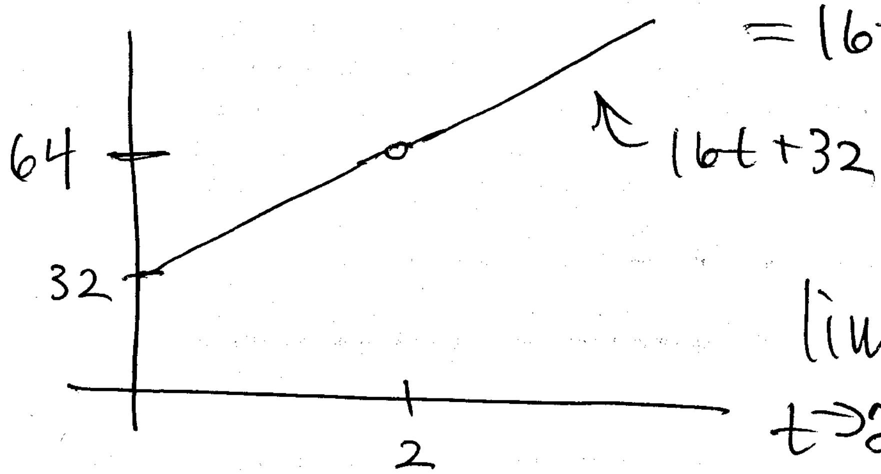
t	3	2.5	2.1	2.05	2	1.95	1.9	1.5	1
$f(t)$	80	72	65.6	64.8	•	63.2	62.4	56	48

Can evaluate this limit for certain:

Algebra: $\underline{16\cdot\frac{t^2-4}{t-2}} = 16\cdot\frac{(t+2)\cancel{(t-2)}}{\cancel{(t-2)}}$

$= \underline{16\cdot(t+2)}$ if $t\neq 2$

$= 16t+32$



$$\underline{\lim_{t\to 2} 16\cdot\frac{t^2-4}{t-2} = 64}$$