6-13-2013

Final Exam – Thurs 6/20 $10^{30} - 1^{15}$

Cumulative. Disproportionate coverage of 11.1, 11.2, 11.4. Calculator allowed. 5 3x5 cards with notes <u>or</u> 1 8½x11 sheet

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<u>11.1</u> Difference Equations I.

y_n = value of some quantity at <u>"time n"</u>

<u>Idea</u>: Suppose that y_n depends in some way on y_{n-1}. Can we find a formula for y_n or otherwise analyse it?

$\boxed{\text{e.g 2}}$ y_n = balance in account after n years.

$y_0 = 40$ (initial amount).

$$y_1 = \underbrace{40 + (.06)(40)}_{(1.06)(40)} - 3 = (1.06)(40) - 3$$

$$y_2 = (1.06)[(1.06)(40) - 3] - 3$$

$$y_3 = (1.06)[(1.06)[(1.06)(40) - 3] - 3] - 3$$

$\vdots$

Easier: $y_0 = 40$

$$y_1 = (1.06)\,y_0 - 3$$

$$y_2 = (1.06)\,y_1 - 3$$

$$y_3 = (1.06)\,y_2 - 3$$

$$\vdots$$

$$y_n = (1.06)\,y_{n-1} - 3$$

Q: What is the balance at any time n?
What happens to balance in long run?
Does it run out? If so, when?
If it grows, ~~can~~ how much more can't withdraw?

Try some terms:

$$y_0 = 40$$

$$y_1 = (1.06)(40) - 3 = 39.40$$

$$y_2 = (1.06)(39.40) - 3 = 38.76$$

$$y_3 = (1.06)(38.76) - 3 = 38.08$$

$$y_4 = 37.36$$

.60

.64

.68

.72

A difference equation looks like:

$$y_n = a y_{n-1} + b \qquad y_0 = \text{given}$$

$$y_1 = a y_0 + b$$

$$y_2 = a(a y_0 + b) + b = a^2 y_0 + ab + b$$

$$y_3 = a(a^2 y_0 + ab + b) + b$$

$$= a^3 y_0 + a^2 b + ab + b$$

$$= a^3 y_0 + b(1 + a + a^2)$$

$$y_n = a^n y_0 + b\underbrace{(1 + a + a^2 + a^3 + \cdots + a^{n-1})}_{\frac{1-a^n}{1-a}}$$

Why? $1 + a + a^2 + a^3 + \cdots + a^{n-1}$

$$a(1 + a + a^2 + a^3 + \cdots + a^{n-1})$$

$$= a + a^2 + a^3 + a^4 + \cdots + a^{n-1} + a^n$$

$$1 + \cancel{a} + \cancel{a^2} + \cancel{a^3} + \cdots + \cancel{a^{n-1}}$$

$$-\ (\cancel{a} + \cancel{a^2} + \cancel{a^3} + \cancel{a^4} + \cdots + \cancel{a^{n-1}} + a^n)$$

$$= 1 - a^n$$

$$(1+a+a^2+\cdots+a^{n-1})$$

$$-a(1+a^2+\cdots+a^{n-1}) =$$

$$(1-a)(1+a^2+a^3+\cdots+a^{n-1}) = 1-a^n$$

$$(1+a^2+a^3+\cdots+a^{n-1}) = \frac{1-a^n}{1-a}$$

$$y_n = a^n y_0 + b\left(\frac{1-a^n}{1-a}\right)$$

$$= a^n y_0 + b\left(\frac{1}{1-a} - \frac{a^n}{1-a}\right)$$

$$= a^n y_0 + \frac{b}{1-a} - a^n\left(\frac{b}{1-a}\right)$$

$$\boxed{y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n \quad (a \neq 1)}$$

$$y_n = a y_{n-1} + b$$

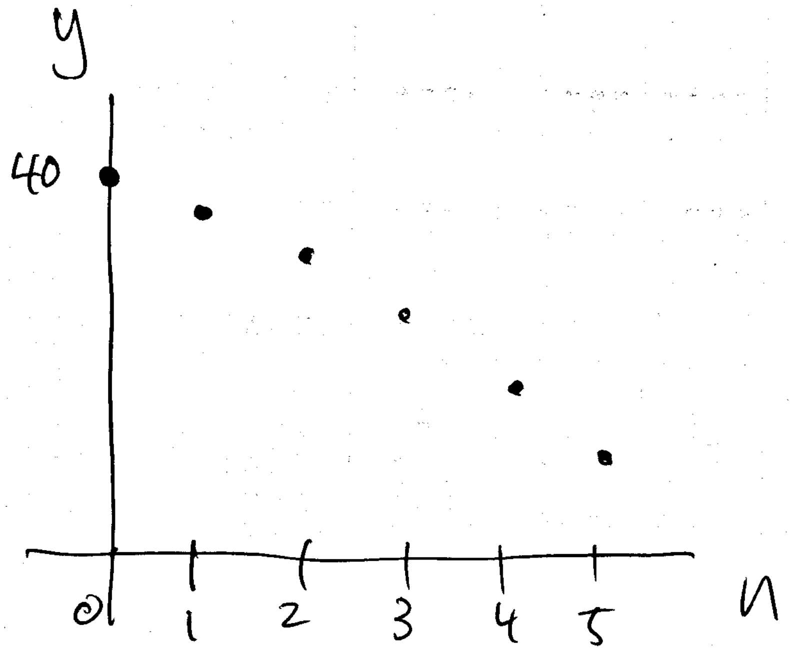
e.g. $y_n = (1.06)y_{n-1} - 3 \qquad a = 1.06 \quad b = -3$

$$y_n = \frac{-3}{(1-1.06)} + \left(40 - \frac{-3}{(1-1.06)}\right)(1.06)^n$$

$$= \frac{-3}{-.06} + \left(40 - \frac{-3}{-.06}\right)(1.06)^n$$

$$= 50 + (40 - 50)(1.06)^n$$
$$= 50 - 10(1.06)^n$$

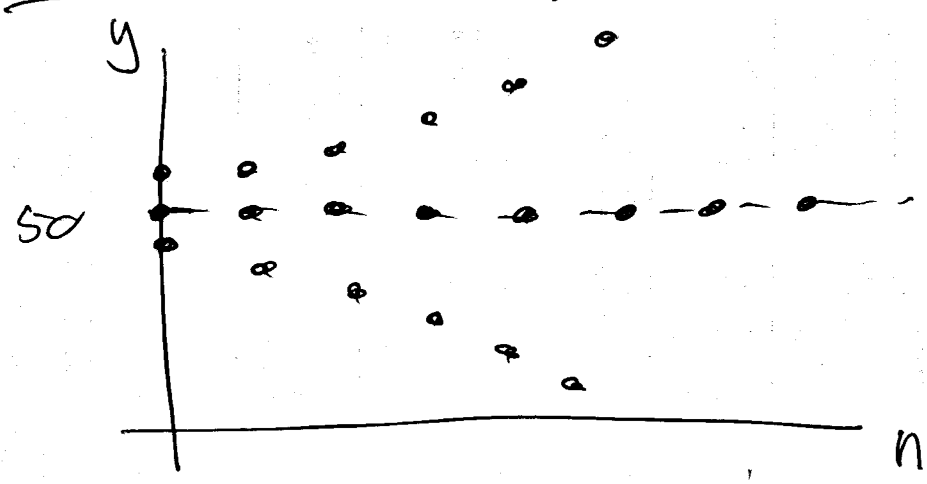
$(1.06)^2 \approx 1.12$
$(1.06)^3 \approx 1.19$
$(1.06)^4 \approx 1.26$
$(1.06)^5 \approx 1.34$
$\vdots$



$$y_n = (1.06)\,y_{n-1} - 3 \qquad y_0 = 40$$

$$y_n = 50 + (y_0 - 50)(1.06)^n$$

Q: What is significant about 50?



$$\boxed{50 = \frac{b}{1-a}}$$

#2. $y_n = -3y_{n-1} + 16$ $a = -3$ $b = 16$

$$\frac{b}{1-a} = \frac{16}{1-(-3)} = \frac{16}{4} = 4$$

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n$$

$$y_n = 4 + (y_0 - 4)(-3)^n$$

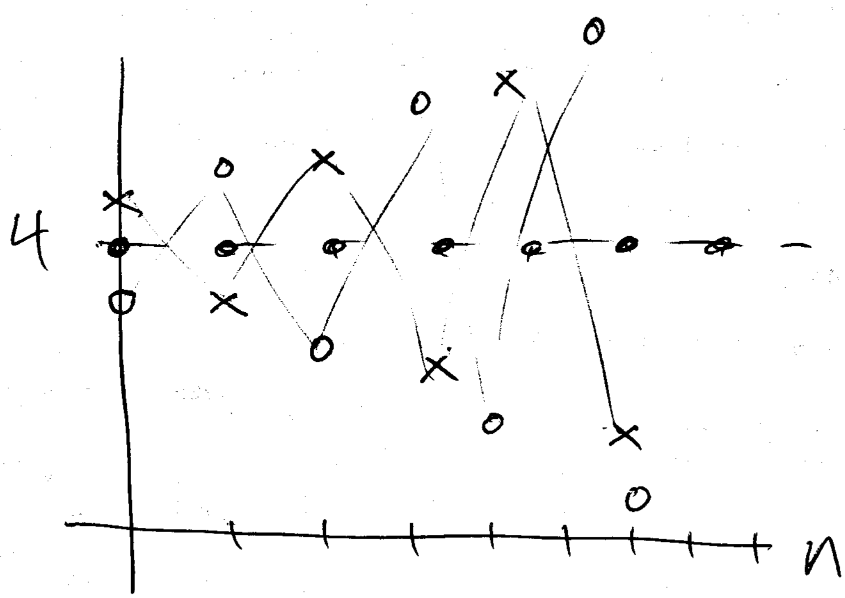
$(-3)^1 = -3$

$(-3)^2 = 9$

$(-3)^3 = -27$

$(-3)^4 = 81$

$(-3)^5 = -243$



4. $y_n = \frac{1}{3} y_{n-1} + 4$ $\quad a = \frac{1}{3} \quad b = 4$

$$\frac{b}{1-a} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}}$$

$$= 4 \cdot \frac{3}{2} = 6$$

$$y_n = 6 + (y_0 - 6)\underbrace{\left(\frac{1}{3}\right)^n}$$

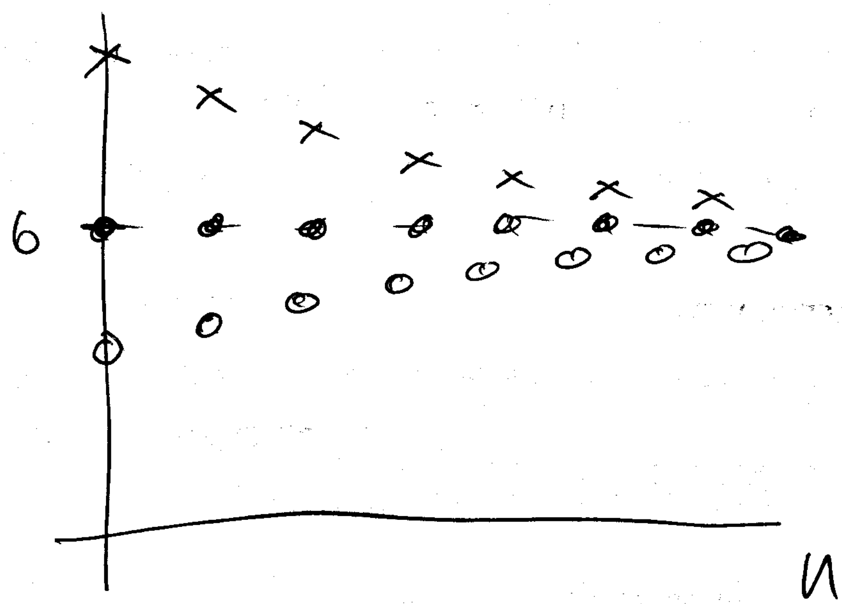
$\left(\frac{1}{3}\right)^1 = .3333$

$\left(\frac{1}{3}\right)^2 = .1111$

$\left(\frac{1}{3}\right)^3 = .0370$

$\left(\frac{1}{3}\right)^4 = .0123$

$\vdots$



$y_0 < 6$

$(y_0 - 6) < 0$

#24) $y_0 = 55$ $\quad$ y_n = balance after n years.

$$y_1 = (1.2)55 - 36 = 1.2\,y_0 - 36$$

$$y_2 = 1.2\,y_1 - 36$$

$$y_n = 1.2\,y_{n-1} - 36 \qquad a = 1.2 \quad b = -36.$$